SOL GEOMETRY GROUPS ARE NOT ASYNCHRONOUSLY AUTOMATIC

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Abstract. Lattices in the 3-dimensional Lie group Sol are shown to be not asynchronously automatic. The proof consists of an interaction of geometry and logical complexity arguments.

In 1910 Max Dehn formulated the word problem for finitely presented groups, and gave a beautiful algorithm to solve the word problem in the case of closed hyperbolic surface groups. Since then, group theorists have been developing many strategies for solving the word problem for various classes of groups.

More recently, Cannon [C] and Gromov [Gr] have offered substantial generalizations of the class of groups to which Dehn's algorithm applies. Motivated in part by this work, [ECHLPT] introduced the class of automatic groups. A group is called automatic if it can be described in the following sense by primitive computing machines called finite state automata. First, the group has a finite generating set, and there is a finite state automaton which recognizes a set of representative words in the generators for all the group elements. Secondly, there are finite state automata which can tell when two words, which are recognized by the first automaton above, represent elements of the group which are equal or just differ by a generator.

Automatic groups have an efficient algorithm for solving word problem. Certain large classes of groups, such as the hyperbolic groups introduced by Gromov in [Gr], are automatic. However, an automatic structure imposes certain restrictions on the geometry of a group. For instance, an automatic group must have a quadratic isoperimetric inequality. This is how one sees that the Baumslag-Solitar groups are not automatic.

The concept of an asynchronously automatic group [ECHLPT] is an attempt to broaden the class of groups which can be computed by finite state machines. The Baumslag-Solitar groups are asynchronously automatic. One may ask which 3-manifold groups are (asynchronously) automatic. In [ECHLPT] it is shown that the fundamental group of a closed 3-manifold which satisfies Thurston's geometrization conjecture is automatic if and only if none of the prime components of the manifold are modeled on either Nil or Sol. It

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is also shown that Sol manifold groups are not automatic, and that closed Nil manifold groups are not even asynchronously automatic.

If a group is asynchronously automatic, then it satisfies a purely geometric condition: it is asynchronously combable with a departure function. See section 1 for precise definitions. Indeed, one often shows that a group is not (asynchronously) automatic by showing that it fails to be (asynchronously) combable. In [Bri] it is shown that Sol lattices are asynchronously combable with departure. Nevertheless, we can prove the following.

Theorem 1. *Lattices in the 3-dimensional Lie group Sol are not asynchronously automatic.*

By the remarks preceeding the statement of Theorem 1, we see that the proof of Theorem 1 will have to use something more than just the geometry of an asynchronous combing. The extra ingredient used is the fact that the combing associated to an asynchronously automatic structure has to be a regular language. We feel that this interaction of logical complexity and geometry is at the heart of the subject of automatic groups, and should be kept in mind if, for instance, one has a CAT(0) group which one believes to be not automatic. In [Bra] this interaction between geometry and logical complexity is used to prove that there is essentially only one type of asynchronously automatic structure on closed hyperbolic surface groups. In [BBr] this interaction is used to prove that certain HNN's of $\mathbb{Z}+\mathbb{Z}$ which have quadratic isoperimetric inequality are not biautomatic.

Now, we explain the layout of the rest of the paper. Section 1 contains the background on asynchronously automatic structures and asynchronous combings with departure. Section 2 gives a brief introduction to the Lie group Sol, and provides some basic results which are needed in the proof of Theorem 1. We give a presentation of a typical lattice G in Sol, and give an explicit embedding of G in Sol. The remaining sections are devoted to the proof of Theorem 1.

In section 3 we give the basic reduction of the proof. By examining the logical complexity of the asynchronous combings associated to an asynchronously automatic structure, we see that the proof reduces to two very different cases. These are treated in sections 4 and 5, thus completing the proof. We conclude with some remarks and questions in section 6.

§1. Asynchronously automatic groups

This section provides the fundamental concepts of asynchronous automatic structures. These definitions can be found in the literature; for example [ECHLPT] chapters 1 and 2, [BGSS] and [NS].

Let G be a finitely generated group and $A \subset G$ be a finite set of monoid generators. Typically, one takes A to be the union of a finite set of group generators for G and the set of their inverses. Let A^* denote the free monoid on A and $w \mapsto \overline{w}$ denote the natural map $A^* \to G$. Call elements of A^*, *words* over the finite alphabet A. We are interested in subsets $L \subset A^*$ which map onto G.

We can place restrictions on the logical structure of L as follows. A *finite state automaton* with alphabet A, is a finite, directed graph, whose edges are labeled by elements of A

such that no two edges with the same label can have the same initial vertex. We call the vertices of the graph *states*. One state is called the *start state*, and a subset of the states are called *accept states*. Reading the labels along a directed edgepath, with initial vertex the start state and terminal vertex an accept state, gives us an element of A^* called an *accepted word*. A subset $L \subset A^*$ is called a *regular language* if it is the set of accepted words of some finite state automaton with alphabet A. If $L \subset A^* \to G$ is a regular language and maps onto G we say it is a *rational structure* for G.

We can also view L as a set of edgepaths in a metric space, and place geometric restrictions on it. Given a finite monoid generating set $A \subset G$, one can construct the *Cayley graph* $\Gamma_A(G)$ of G with respect to A. The set of vertices of $\Gamma_A(G)$ is just G itself. There is a directed edge from g to ga for each $g \in G$ and $a \in A$. Since A is finite, $\Gamma_A(G)$ is a locally finite graph. Let each edge of $\Gamma_A(G)$ be isometric to the interval $[0,1]$, and define the distance between two points to be the infimum of the lengths of all paths in $\Gamma_A(G)$ joining them. This path metric makes $\Gamma_A(G)$ into a geodesic metric space (i.e. for all $x, y \in \Gamma_A(G)$ there is a path joining x to y of length equal to $d(x,y)$, the distance between x and y). A word $w \in A^*$ gives an edgepath in $\Gamma_A(G)$ (also denoted by w) which starts at the identity and ends at the vertex $\overline{w}$. We parameterize these paths by arc length up to their endpoints, where they remain constant. Thus $w(t) = \overline{w}$ for all $t \geq \text{length}(w)$. Here $\text{length}(w)$ denotes the length of the path w, which equals the length of w as an element of A^*.

We say $L \subset A^*$ has the *K-fellow traveler property* if there exists $K \geq 0$ such that for all $u, v \in L$ with $\overline{u^{-1}v} \in \overline{A} \cup \{\text{id}\}$ then $d(u(t), v(t)) \leq K$, for all $t \geq 0$. This says that if two L-paths end distance ≤ 1 apart in $\Gamma_A(G)$ then points traveling at unit speed along them (starting at the same time at the identity) stay within K of each other. Say that $L \subset A^*$ has the *asynchronous K-fellow traveler property* if there exists $K \geq 0$ such that for all $u, v \in L$ with $\overline{u^{-1}v} \in \overline{A} \cup \{\text{id}\}$ there are monotonic reparameterizations ϕ and ψ (depending on u and v) such that $d(u(\phi(t)), v(\psi(t))) \leq K$ for all $t \geq 0$. This is similar to the previous definition, except that we now allow points to speed up or slow down (but not backtrack) along the L-paths in order to stay within K of each other. We say that $L \subset A^*$ has a *departure function* $D : \mathbb{R}_+ \to \mathbb{R}_+$ if for all $r > 0$, for all $w \in L$ and for all $0 \leq s, t \leq \text{length}(w)$ we have

$$|s - t| > D(r) \Longrightarrow d(v(s), v(t)) > r\,.$$

An *automatic structure* on G is a rational structure $L \to G$ which satisfies the K-fellow traveler property. We say that G is an *automatic group*. An *asynchronous automatic structure* on G is a rational structure $L \to G$ which satisfies the asynchronous K-fellow traveler property, and has a departure function.

Although we have not mentioned a departure function in defining an automatic structure $L \to G$, the language L actually has a linear departure function. This is proven in theorem 3.3.4 of [ECHLPT]. Thus an automatic structure is a special case of an asynchronous automatic structure, where the reparameterization maps are the identity and the departure function is linear. However, in section 7.4 of [ECHLPT] the Baumslag-Solitar groups are shown to be asynchronously automatic but not automatic. Even automatic groups can admit other asynchronous automatic structures which are not automatic structures. The fundamental groups of closed hyperbolic 3-manifolds which fiber over the circle give examples of such groups. See [NS] for details.

In theorem 2.5.1 (resp. theorem 7.3.2) of [ECHLPT] it is shown that every automatic structure (resp. asynchronous automatic structure) $L \to G$ has a substructure $L' \subset L$ *with uniqueness*; that is, $L' \to G$ is an automatic structure (resp. asynchronous automatic structure), and the projection map restricted to L' is bijective.

The geometric content of the definitions of (asynchronous) automatic structure with uniqueness gives rise to the following concepts. By a *synchronous combing* (resp. *asynchronous combing*) on a finitely generated group G we mean a section σ for the projection map $A^* \to G$, such that $\sigma(G) \subset A^*$ satisfies the K-fellow traveler property (resp. asynchronous K-fellow traveler property). We say that σ is a combing *with departure* if $\sigma(G)$ has a departure function. Clearly an asynchronous automatic structure on G gives rise to an asynchronous combing with departure.

Let $L \to G$ be an (asynchronous) automatic structure with uniqueness, such that the empty word in L maps to $\mathrm{id} \in G$. The words of L (viewed as paths in the Cayley graph of G) have an obvious geometric constraint; namely, they satisfy the (asynchronous) K-fellow traveler property. Given $u, v \in L$ with $d(\overline{u}, \overline{v}) = R$, we can apply the definition of asynchronous K-fellow traveler property to the $(R+1)$ L-paths whose endpoints are the vertices on a geodesic segment from $\overline{u}$ to $\overline{v}$, to conclude that u and v must lie in KR-Hausdorff neighborhoods of each other. Case I of Theorem 1 is proven by exhibiting L-paths which do not satisfy this Hausdorff neighborhood condition. We will say that such paths contradict the asynchronous K-fellow traveler property of L.

In the two lemmas below we list some other geometric properties of words in L, which are derived from the regularity of L. Let q be the number of states in the automaton accepting L. We will denote by $d(\ ,\ )$ the metric in the Cayley graph $\Gamma_A(G)$. Lemma 1.1 says that every subsegment of a word in L will look something like a word in L.

Lemma 1.1. *If $u \in L$ and v is any subsegment of u, then there are words v_1, $v_2 \in A^*$ of length $\leq q$ such that $v_1 v v_2 \in L$.*

Proof. The endpoints of v break u into three segments. Consider u as a directed path in the finite state automaton graph. Let s_1, s_2 be the states at the beginning and end of v respectively. The first segment of u is a path from the start state to s_1; choose a shortest such path. This gives a word in A^* which we call v_1. Similarly, the last segment of u is a path from s_2 to an accept state. Take a shortest such path to get a word v_2. Clearly v_1, v_2 are both of length $\leq q$. By construction, $v_1 v v_2$ is accepted by the automaton, and so is a word in L. □

We will refer to v_1 as a *head* and to v_2 as a *tail*, which we attach to v. Combining this with the (asynchronous) K-fellow traveler property gives a bound on how much L words can back track.

Lemma 1.2. *If $u \in L$ and v a subsegment of u with endpoints a, b then*

$$v \subset B_{K(2q+d(a,b))+q}(a)$$

where $B_r(a)$ denotes the ball of radius r about a in $\Gamma_A(G)$.

Proof. Lemma 1.1 ensures that we can find short words $v_1, v_2 \in A^*$ so that $w = v_1 v v_2 \in L$. Then $d(\mathrm{id}, \overline{w}) \leq 2q + d(a, b)$. Now the empty word in L maps to id so that the (asynchronous) K-fellow traveler property gives

$$w \subset B_{K(2q+d(a,b))}(\mathrm{id})$$

Thus

$$w \subset B_{K(2q+d(a,b))+q}(\overline{v_1})$$

and so

$$v \subset B_{K(2q+d(a,b))+q}(a)$$

as required. □

§2. Sol geometry groups

The Lie group Sol is one of the eight basic geometries that one considers in the study of 3-manifolds. There is a good collection of background literature on Sol: for example [G] chapter 3, and [Sc]. We give the definition and describe basic properties of Sol here, mainly as a means of establishing notation for the remainder of the paper.

One can describe Sol as a semi direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the map

$$(x, y) \mapsto (e^t x, e^{-t} y)\,.$$

We can identify Sol with $\mathbb{R}^3$ so that the normal subgroup $\mathbb{R}^2$ maps to the xy-plane. Multiplication is defined by

$$(x, y, z)(x', y', z') \;=\; (x + e^{-z}x', y + e^{z}y', z + z')\,.$$

Note that conjugation by $(0, 0, t)$

$$(0, 0, -t)(x, y, 0)(0, 0, t) \;=\; (e^t x, e^{-t} y, 0)$$

acts on the xy-plane in exactly the same way as conjugation by t acts on $\mathbb{R}^2$. Thus, $t \mapsto (0, 0, t)$ preserves the correct action of $\mathbb{R}$ on $\mathbb{R}^2$. Using the group multiplication we obtain a left invariant metric of the form

$$ds^2 \;=\; e^{2z}dx^2 + e^{-2z}dy^2 + dz^2\,.$$

Let $d_S(\ ,\ )$ denote the distance function on Sol. The distance between two points is given by $\inf\{\text{length}(\gamma)\}$ where γ ranges over all rectifiable paths between the points. Since Sol is a complete Riemannian manifold, this inf is realized by a geodesic path in Sol.

Consider two points in the xz-plane. The infimum above is achieved for a path γ which lies entirely in this plane (since the $e^{-2z}dy^2$ term in ds^2 will contribute a positive amount to an arclength integral for paths with nonzero y component). Thus the metric on the xz-plane induced by $ds^2 = e^{2z}dx^2 + dz^2$ agrees with the Sol metric. On mapping this plane to the upper half plane

$$\{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 > 0\}$$

by

$$(x, z) \mapsto (x_1, x_2) = (x, e^{-z})$$

the metric becomes the standard hyperbolic metric

$$ds^2 = \frac{{dx_1}^2 + {dx_2}^2}{{x_2}^2}.$$

Thus the xz-plane is a convexly embedded copy of $\mathbb{H}^2$ in Sol. Similarly, the yz-plane is another convex $\mathbb{H}^2$ in Sol. A standard computation in $\mathbb{H}^2$ yields

$$\begin{aligned} d_S((x,0,0),(x',0,0)) &= d_{\mathbb{H}^2}((x,1),(x',1)) \\ &= 2\log\left[\sqrt{1 + ((x-x')/2)^2} + |x-x'|/2\right]. \end{aligned}$$

Thus, for $|x - x'| \geq 2$ we have

$$2\log|x-x'| \leq d_S((x,0,0),(x',0,0)) \leq 2\log\left[\frac{3|x-x'|}{2}\right] \tag{2.1}$$

Note that $|x - x'|$ is the Euclidean distance in the xy-plane from $(x, 0, 0)$ to $(x', 0, 0)$. Similarly, for $|y - y'| \geq 2$ we have

$$2\log|y-y'| \leq d_S((0,y,0),(0,y',0)) \leq 2\log\left[\frac{3|y-y'|}{2}\right] \tag{2.2}$$

We will need to compare Euclidean distances between points in the xy-plane with Sol distances.

Definition 2.1. The two points $(p_1, q_1, 0)$ and $(p_2, q_2, 0)$ are said to have *difference vector* $\langle d_1, d_2\rangle = \langle p_2 - p_1, q_2 - q_1\rangle$.

Lemma 2.2. *If points $(p_1, q_1, 0)$ and $(p_2, q_2, 0)$ in Sol have difference vector $\langle d_1, d_2\rangle$ with $|d_i| \geq 2$ then*

$$\begin{aligned}\max\{2\log|d_1|\,,\ 2\log|d_2|\} &\leq d_S((p_1,q_1,0),(p_2,q_2,0))\\ &\leq 2\log\left[\frac{3|d_1|}{2}\right] + 2\log\left[\frac{3|d_2|}{2}\right]\end{aligned}$$

Proof. Since left multiplication by $(-p_2, q_1, 0)$ is an isometry of Sol, we have

$$\begin{aligned}d_S((p_1,q_1,0),(p_2,q_2,0)) &= d_S((-d_1,0,0),(0,d_2,0))\\ &\geq \max\{d_S((0,0,0),(0,d_2,0))\,,\ d_S((-d_1,0,0),(0,0,0))\}\\ &\geq \max\{2\log|d_2|\,,\ 2\log|d_1|\}\end{aligned}$$

The first inequality is a consequence of the fact that the minimum Sol distance from $(-d_1, 0, 0)$ to the line $x = z = 0$ is precisely $d_S((0,0,0),(-d_1,0,0))$ (and similarly for $(0, d_2, 0)$ and the line $y = z = 0$). The second inequality follows from the estimates (2.1) and (2.2).

By the triangle inequality we also have an upper bound

$$\begin{aligned}d_S((p_1,q_1,0),(p_2,q_2,0)) &= d_S((-d_1,0,0),(0,d_2,0))\\ &\leq d_S((-d_1,0,0),(0,0,0)) + d_S((0,0,0),(0,d_2,0))\\ &\leq 2\log\left[\frac{3|d_1|}{2}\right] + 2\log\left[\frac{3|d_2|}{2}\right]\end{aligned}$$

and so the lemma follows. □

The next lemma gives a computation for the distance from a point in Sol to the xz-plane. This will be useful in section 4. A similar result holds for distances from points to the yz-plane, only z must be replaced by $-z$ in the right hand side of the equation in Lemma 2.3. Let $\mathrm{dist}_S(\ ,\ )$ denote the Sol distance from a point to a closed subset.

Lemma 2.3. *For each point (x, y, z) in Sol we have*

$$\mathit{dist}_S((x,y,z)\,,\ xz\text{-}\mathit{plane}) = \log\left[\sqrt{1+\left(\frac{ye^{-z}}{2}\right)^2} + \frac{|y|e^{-z}}{2}\right]$$

Proof. Since left multiplication by $(-xe^z, 0, -z)$ is an isometry of Sol which preserves the xz-plane we have

$$\begin{aligned}\mathrm{dist}_S((x,y,z)\,,\ xz\text{-plane}) &= \mathrm{dist}_S((0,ye^{-z},0)\,,\ xz\text{-plane})\\ &= \mathrm{dist}_S((0,ye^{-z},0)\,,\ z\text{-axis})\\ &= \mathrm{dist}_{\mathbb{H}^2}((ye^{-z},1)\,,\ x_2\text{-axis})\\ &= \log\left[\sqrt{1+\left(\frac{ye^{-z}}{2}\right)^2} + \frac{|y|e^{-z}}{2}\right]\end{aligned}$$

The second equality is just the observation that any point which minimizes the distance from $(0, ye^{-z}, 0)$ to the xz-plane must have zero x-component (any path γ realizing this distance cannot have a non zero x-component, since this would contribute a positive amount to the arc length integral). The other two equalities follow exactly as in the derivation of estimate (2.1). $\square$

In [Sc], it is shown that any lattice $G < \mathrm{Sol}$ (i.e. discrete, cofinite volume subgroup) is actually uniform (i.e. cocompact) and that the 3-dimensional manifold Sol/G is a torus bundle over S^1 with Anosov monodromy. Thus G has a presentation

$$G = G_\phi = \langle\, a, b, t : [a,b] = 1,\ t^{-1}at = \phi(a),\ t^{-1}bt = \phi(b)\,\rangle \tag{2.3}$$

where $\phi \in \mathrm{Sl}_2(\mathbb{Z})$ is a hyperbolic element (i.e. $|\mathrm{tr}(\phi)| > 2$). Note that ϕ will have two irrational eigenvalues λ and λ^{-1}, with irrational slope eigendirections. In [Bri] it is shown that the groups G_ϕ are asynchronously combable with departure. It is interesting to note that the combing lines of the asynchronous combing provided in [Bri] 'intersect' the base $\langle a, b\rangle$ coset in a pattern which has irrational slopes, and so cannot come from a regular language in the generating set $\{a^{\pm 1}, b^{\pm 1}, t^{\pm 1}\}$.

We can realize $G = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ explicitly as a subgroup of Sol as follows. First embed the normal subgroup $\mathbb{Z}^2$ in $\mathbb{R}^2$ as the standard integer lattice. Now $\phi \in \mathrm{Sl}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ (preserving $\mathbb{Z}^2$) with unit λ-eigenvector (e_1, e_2) and unit λ^{-1}-eigenvector (f_1, f_2). Let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which takes (e_1, e_2) to $(1, 0)$ and (f_1, f_2) to $(0, 1)$. Now define $f : G \to \mathrm{Sol}$ by

$$(a^m, b^n, t^p) \mapsto (U(m,n), p\log(\lambda))\,. \tag{2.4}$$

It is clear from the definition of multiplication in Sol given above that f is a monomorphism and that its image is a lattice in Sol. There is an obvious way to extend f to a G-equivariant map $f' : \Gamma_A(G) \to \mathrm{Sol}$. For example, we may map the edge $[\mathrm{id}, t]$ to the interval $[(0,0,0), (0,0,\log(\lambda))]$ on the z-axis. Map the edge $[\mathrm{id}, a]$ (resp. $[\mathrm{id}, b]$) to the straight segment $[(0,0,0), (U(1,0), 0)]$ (resp. $[(0,0,0), (U(0,1), 0)]$) in the xy-plane. Finally, we map the other edges of $\Gamma_A(G)$ to the appropriate G translates of these three segments. Since both $\Gamma_A(G)/G$ and Sol/G are compact, theorem 1 of [C] shows that there are constants $\epsilon_1, \epsilon_2 > 0$ such that

$$\epsilon_1 d_S(f'(x), f'(y)) \le d_A(x, y) \le \epsilon_2 d_S(f'(x), f'(y)) \tag{2.5}$$

for all $x, y \in \Gamma_A(G)$. This estimate enables us to translate problems about the geometry of L-paths in $\Gamma_A(G)$ into questions about paths in Sol.

§3. Proof of Theorem 1: Reduction

We prove Theorem 1 by contradiction. Suppose we are given that G is asynchronously automatic. From section 1 we know that there is an asynchronous automatic structure $L \subset A^*$ with uniqueness for G, where $A = \{a, b, t, a^{-1}, b^{-1}, t^{-1}\}$ is the standard monoid generating set corresponding to the presentation (2.3). Throughout this proof Δ denotes the finite state automaton which accepts L. Recall that Δ is a finite directed graph, with edges labeled by the elements of A. Let q denote the number of states in Δ.

Definition 3.1. A *loop* in Δ is a closed directed edgepath (not necessarily a simple closed path) contained in Δ.

Given a loop with a base vertex one can read off a word in A^*, which will have a positive, negative, or zero t-exponent sum. We say that the loop has (respectively) positive, negative, or zero *type*. Note that the type of a loop is independent of the base vertex chosen.

Definition 3.2. A state s of Δ is said to be

(1) *accessible* if there is a directed edgepath from the start state to s, and
(2) *live* if there is a directed edgepath from s to an accept state.

The proof of Theorem 1 breaks into two cases, depending on the structure of the graph Δ.

Case I: There are at least two types of loops based at some live and accessible state s in Δ.

We may assume that on reading two of the loops at s one obtains the words

$$l_1 = t^l w = t^l a^{m_1} b^{n_1} \tag{3.1}$$

and

$$l_2 = v = a^{m_2} b^{n_2} \tag{3.2}$$

where $0 < l \in \mathbb{Z}$ and $m_i, n_i \in \mathbb{Z}$ are such that $v \neq \mathrm{id} \in \langle a, b \rangle \cong \mathbb{Z}^2$.

Why can we make this assumption? First, we can ensure that one loop is of zero type by taking appropriate combinations of positive and negative type loops if necessary. Secondly, we may assume that the loop corresponding to l_1 is of positive type (i.e. $l > 0$), since the geometry of the negative-zero case is similar to this. This will become apparent on reading the arguments for the positive-zero case in section 4.

Finally, the two loops at s may give words $l'_1, l'_2 \in A^*$ with $\overline{l'_i} = \overline{l_i}$ but $l'_i \neq l_i$. Let M denote the larger of the lengths of the l'_i. Then every L-path stays within an $M/2$-neighborhood of the corresponding edgepath constructed using the l_i instead of the l'_i. In section 4 we shall consider the subset of L consisting of words of the form $u_1 w u_2$, where w is a word in the free monoid on $\{l'_1, l'_2\}$. Here u_1 (resp. u_2) is obtained by tracing a shortest directed edgepath from the start state to s (resp. from s to an accept state), so that u_1, u_2 have length at most q. These L-words track the left translate by $\overline{u_1}$ of the submonoid of G generated by $\{\overline{l_1}, \overline{l_2}\}$. In section 4 we study the geometry of such submonoids of G. Certain paths in these monoids diverge far apart in $\Gamma_A(G)$ and then re-converge, so that the corresponding L-paths contradict the asynchronous K-fellow traveler property (for any K). Actually, we will work with the free monoid on $\{l_1, l_2\}$, but since L-paths stay within an $M/2$ neighborhood of these paths the contradiction still holds.

Remark 3.3. This is an example of how an asynchronous automatic structure $L \to G$ is more restrictive than an asynchronous combing with departure. Since L is regular, it must be accepted by some finite state automaton Δ. Reading off a set of n based loops at some live and accessible state of Δ gives words $w_1, \dots, w_n \in A^*$. As above, certain L-words must track (a left translate of) the submonoid of G generated by $\{\overline{w_1}, \dots, \overline{w_n}\}$. This involves the group multiplication on G, unlike an asynchronous combing with departure which is a purely geometric notion. In order to study the shape of an asynchronous automatic structure on a group G, we need an understanding of the geometry of finitely generated submonoids of G. We examine very specific two generator submonoids of a Sol lattice in section 4, and consider Hausdorff neighborhoods of single generator submonoids in section 5.

Another example of this interaction of logical complexity and geometry is seen, for example, in the case of torsion free hyperbolic groups, where where single generator submonoids track geodesic paths in the Cayley graph. This fact was an essential ingredient in the proof of Theorem 1 of [Bra].

Case II: There is at most one type of loop at each live and accessible state s in Δ.

We use the asynchronous K-fellow traveler property to show that L cannot map onto G. In particular there is an $R > 0$ and $w \in L$ such that $\overline{L}$ does not contain all the points of G that are within R of $\overline{w}$;

$$G \cap B_R(\overline{w}) \not\subset \overline{L}. \tag{3.3}$$

This gives a contradiction in case II and proves Theorem 1. This case includes the case where L is a *simply starred language*. This means that there is at most one loop at each live and accessible state of the finite state automaton accepting L. We note that in the simply starred case the language L has only got polynomial growth, and so could not surject onto G. We say that in Case II L is *geometrically simply starred*: that is, it is simply starred with respect to the type of loops in Δ.

We shall see in section 5 that L-words have a particularly simple geometric shape in Sol: they 'change direction' only finitely many times. The argument in section 5 is similar in flavor to previous work on simply starred languages, in particular the work in [NS] on asynchronous automatic structures on virtually abelian groups, and the work of [BG] on nilpotent groups.

The L-word w with the property (3.3) has a special form. The following definition establishes the terminology and concepts which will be used in section 5 to construct w and prove the neighborhood property (3.3).

Definition 3.4. An *L-track* is a directed edgepath in Δ beginning at the start state and ending at an accept state. Every L-word, by definition, describes an L-track in Δ.

A *stem* is an L-track which has no repeated states. Thus the length (i.e. number of edges) of a stem is at most $q - 1$. Every L-track 'contains' (as a subset of Δ) at least one stem; obtained by just omitting any loops that may occur as we proceed along the L-track.

All states along an L-track are live and accessible. Define a live and accessible state s to be *positive* (resp. *negative*, *zero*) if there is a loop in Δ based at s which is of positive

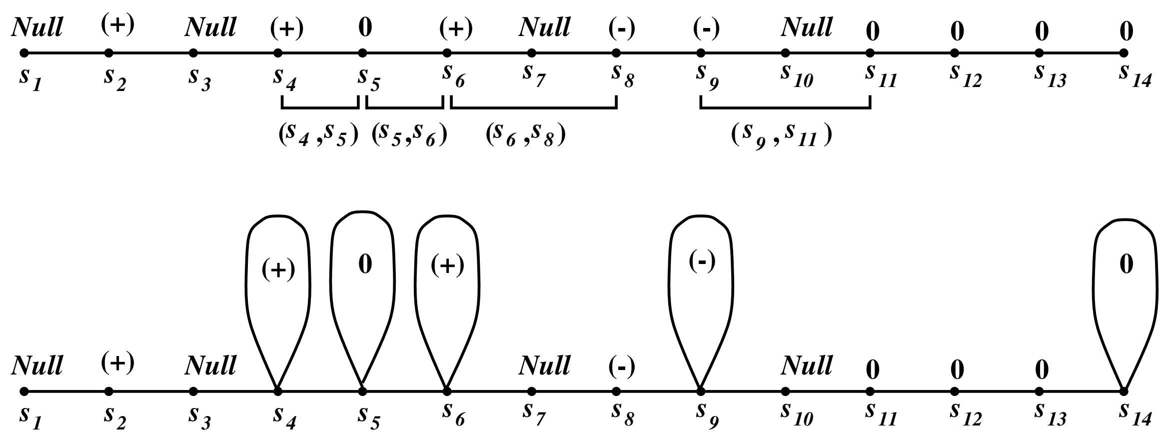

Figure 1. A maximal stem and an associated standard L-track.

(resp. negative, zero) type. If there are no loops based at s we call it a *null* state. The hypothesis of Case II ensures that these notions are well defined.

Definition 3.5. A stem induces a linear order on its set of states. Label these states by $s_1, \ldots, s_m$ where the subscripts denote the ordering. We call (s_i, s_j) where $i < j$ a *change* if it has the following properties:

1. the set $\{s_{i+1}, \ldots, s_{j-1}\}$ is either empty or consists entirely of null states,
2. neither of the states s_i or s_j are null, and
3. the states s_i and s_j are not both positive, negative or zero.

For example, the stem in figure 1 has 4 changes: namely, (s_4, s_5), (s_5, s_6), (s_6, s_8) and (s_9, s_{11}).

Since Δ is a finite graph, it can contain only finitely many stems.

Definition 3.6. A stem is said to be *maximal* if it contains a maximal number of changes. We call $w \in L$ a *maximal L-word* if its L-track contains a maximal stem.

For each change (s_i, s_j) along a maximal stem add a loop of the appropriate type to the vertex s_i. Also add a loop to the final non-null vertex of the stem. We call the resulting track a *standard maximal L-track*. See figure 1. If a word has a standard maximal L-track as its track we call it a *standard maximal L-word*.

In section 5 we will work with these standard maximal L-words to prove that there is a word $w \in L$ which satisfies the neighborhood property (3.3), and hence that L does not map onto G.

§4 Proof of Theorem 1: Case I

Recall from section 3 that, in this case, there is a positive type loop l_1 and a type zero loop l_2, based at a live and accessible state s in the finite state automaton graph Δ for L.

We assume that l_1 and l_2 are of the forms described in (3.1) and (3.2) respectively.

In order to obtain a contradiction in this case we need to find two words γ_1 and γ_2 in the free monoid on $\{l_1, l_2\}$ and a constant $B > 0$ such that $d(\overline{\gamma_1}, \overline{\gamma_2}) \leq \frac{B}{K} - 2q$ yet the path γ_1 is not contained in a $(B+q)$-Hausdorff neighborhood of γ_2. We know from Lemma 1.2 that there are L-words $v_i = u_1\gamma_i u_2$ where the lengths of the u_i are bounded by q, the number of states in Δ. Thus $d(\overline{v_1}, \overline{v_2}) < \frac{B}{K}$ but v_1 does not lie inside a B-Hausdorff neighborhood of v_2. This contradicts the asynchronous K-fellow traveler property of L.

We may suppose that the γ_i are paths in Sol, provided we use the estimate (2.5) to translate the bounds. There is a constant $C > 0$ such that γ_1 is not contained in a C Hausdorff neighborhood of γ_2 in Sol, and yet

$$d_S(\overline{\gamma_1}, \overline{\gamma_2}) \ \leq \ \alpha_1 C - \alpha_2 \tag{4.1}$$

where $C + \frac{B+q}{\epsilon_1}$, $\alpha_1 = \frac{\epsilon_1}{\epsilon_2 K}$ and $\alpha_2 = \frac{(2K+1)q}{\epsilon_2 K}$.

It is instructive to begin by considering the special case where $l_1 = t^l$ and $l_2 = v = a^{m_2}b^{n_2}$ (i.e. $w = \mathrm{id} \in \langle a, b\rangle$). We shall consider the general case of l_i given by (3.1) and (3.2) in the Remarks 4.4 at the end of this section. Recall that the embedding f defined by (2.4) takes the eigendirections of ϕ to the x- and y-axes in Sol. Since these eigendirections have irrational slopes, the element v corresponds to a point $(x, y, 0)$ in Sol where y/x is irrational. Thus, the vector v is not parallel to either axis. For the sake of concreteness, we assume that both x and y are positive.

Definition/Remark 4.1. For $k \in \mathbb{Z}$ call the left translate of the xy-plane by $t^k = (0, 0, k\log(\lambda))$ the *k-level*. The coset $t^k\langle a, b\rangle$ lies in the k-level. If P is a point in a k-level and Q is a point in a $(k+n)$-level, then an easy arc length computation gives $d_S(P, Q) \geq n\log(\lambda)$. This fact will be used later to show that the paths γ_i diverge.

We construct the paths γ_1 and γ_2 in a piecewise fashion. There are two main steps; a divergence estimate, and a convergence construction. Here is an outline of the procedure.

(1) **The divergence step:** The initial segments $[\gamma_1]_1$ and $[\gamma_2]_1$ are defined so that the paths diverge. The definitions involve Nth powers of the l_i where $N \in \mathbb{N}$ is large and even. The path γ_2 is of the form $[\gamma_2]_1 l_1^{\frac{N}{2}} u$ where u is some word in the monoid $\langle l_1, l_2\rangle$. We prove that the endpoint E_1 of $[\gamma_1]_1$ lies outside of a $C(N)$-neighborhood of all of γ_2, for large N. The bound $C(N)$ is a linear function of N.

(2) **The convergence construction:** We iterate a construction which *forces the γ_i to converge*. Suppose that the endpoints E_j and F_j of $[\gamma_1]_j$ and $[\gamma_2]_j$ lie on some k_j-level. We add segments to the $[\gamma_i]_j$ to obtain paths $[\gamma_1]_{j+1}$ and $[\gamma_2]_{j+1}$ with endpoints E_{j+1} and F_{j+1} on some k_{j+1}-level. Under suitable conditions distances decrease according to

$$d_S(E_{j+1}, F_{j+1}) \leq \frac{2}{3} d_S(E_j, F_j)\,.$$

We find lower bounds for N which ensure that this process continues until setting

$$\gamma_i = [\gamma_i]_r$$

for some r, we have

$$d_S(\overline{\gamma_1}, \overline{\gamma_2}) \leq \alpha_1 C(N) - \alpha_2$$

which is the required contradiction.

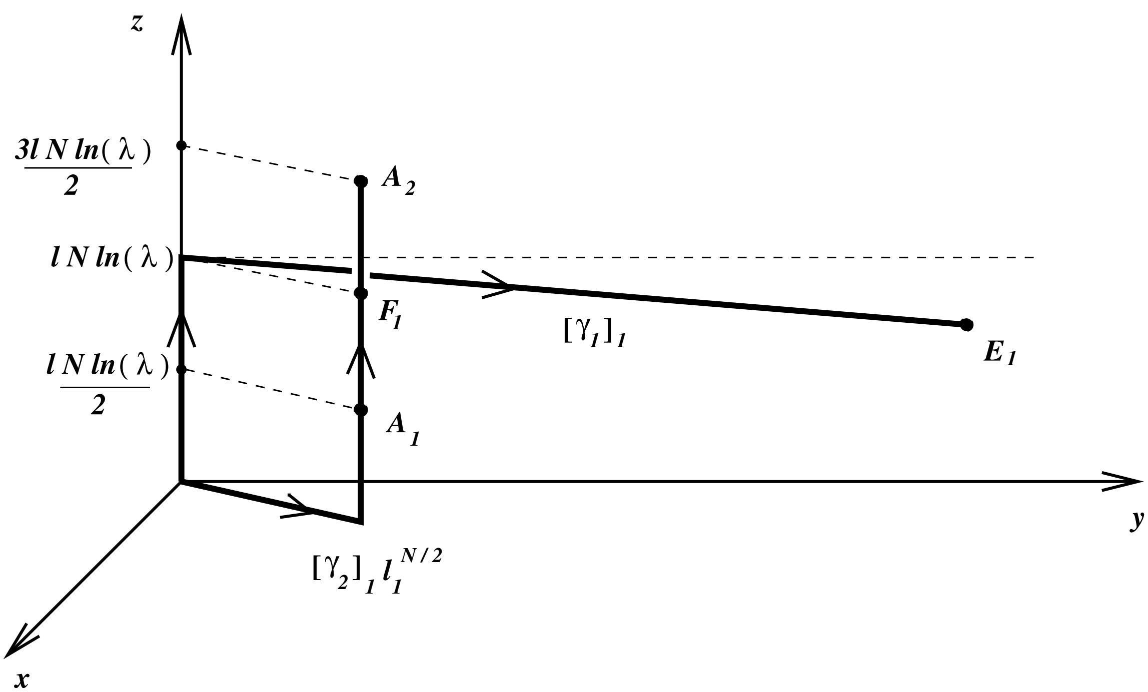

FIGURE 2. The segments $[\gamma_1]_1$ and $[\gamma_2]_1 l_1^{\frac{N}{2}}$ in Sol.

Step 1.(Divergence) For $N \in \mathbb{N}$ large and even define

$$[\gamma_1]_1 = l_1^N l_2^{\lambda^{lN} + \beta_1}$$

and

$$[\gamma_2]_1 = l_2 l_1^N$$

where $\lambda^{2l}+1 \leq \beta_1 < \lambda^{2l}+2$ is chosen such that $\lambda^{lN}+\beta_1 \in \mathbb{N}$. Figure 2 is a picture of the paths $[\gamma_1]_1$ and $[\gamma_2]_1 {l_1}^{\frac{N}{2}}$. Let E_1 be the endpoint $\overline{[\gamma_1]}_1$ and F_1 be the endpoint $\overline{[\gamma_2]}_1$. Thus

$$\begin{aligned}
E_1 &= \overline{[\gamma_1]}_1 \\
&= (0,\, 0,\, lN\log(\lambda))\;\big((\lambda^{lN}+\beta_1)x,\, (\lambda^{lN}+\beta_1)y,\, 0\big) \\
&= \big([1+\beta_1\lambda^{-lN}]x,\, [\lambda^{lN}+\beta_1]\lambda^{lN}y,\, lN\log(\lambda)\big)
\end{aligned}$$

and

$$F_1 \;=\; (x,\, y,\, lN\log(\lambda))\,.$$

The points

$$\begin{aligned}
A_1 &= \left(x,\, y,\, \frac{lN}{2}\log(\lambda)\right) \\
A_2 &= \left(x,\, y,\, \frac{3lN}{2}\log(\lambda)\right)
\end{aligned}$$

both lie on $[\gamma_2]_1 {l_1}^{\frac{N}{2}}$. The next lemma tells us that E_1 lies outside a large neighborhood of γ_2.

Lemma 4.2. *Let E_1, $[\gamma_1]_1$ and $[\gamma_2]_1$ be as above. Suppose that γ_2 is formed by adjoining ${l_1}^{\frac{N}{2}}u$ to the endpoint of $[\gamma_2]_1$, where u is any word in the monoid $\langle l_1, l_2\rangle$. Then, if*

$$N \geq \max\left\{\frac{2\log_\lambda(2)}{l}\,,\; \frac{2\log_\lambda(y)}{l}\right\}$$

E_1 lies outside a $C(N)$-neighborhood of γ_2 where

$$C(N) \;=\; \frac{Nl}{2}\log(\lambda)$$

Proof. The subsegment of $[\gamma_2]_1$ from $(0,0,0)$ to A_1 lies below the $\frac{lN}{2}$-level and so E_1 is at least $\frac{lN}{2}\log(\lambda)$ from it. Similarly, the last segment of γ_2 (from A_2 to the endpoint $\overline{\gamma_2}$) lies on or above the $\frac{3lN}{2}$-level, and so E_1 is at least $\frac{lN}{2}\log(\lambda)$ from it too.

So we only need to estimate the distance from E_1 to $[A_1, A_2]$, the middle segment of γ_2. For $P=(x,y,z_p)\in[A_1,A_2]$ we have

$$\frac{lN}{2}\log(\lambda) \leq z_p \leq \frac{3lN}{2}\log(\lambda)\,.$$

Thus, Lemma 2.3 gives the upper bound

$$\begin{aligned}
\operatorname{dist}_S(P, xz\text{-plane}) &= \log\left[\sqrt{1+\left(\frac{ye^{-z_p}}{2}\right)^2} + \frac{ye^{-z_p}}{2}\right] \\
&\leq \log\left[\sqrt{1+\left(\frac{y}{2\lambda^{\frac{lN}{2}}}\right)^2} + \frac{y}{2\lambda^{\frac{lN}{2}}}\right] \\
&\leq \log(2)
\end{aligned}$$

for $N \geq N_1 = \frac{2}{l}log_\lambda(y)$. Lemma 2.3 also gives a lower bound for the distance from E_1 to the xz-plane.

$$\begin{aligned}
&\operatorname{dist}_S(E_1\,,\, xz\text{-plane}) \\
&= \operatorname{dist}_S\left(\left([1+\beta_1\lambda^{-lN}]x,\, [\lambda^{lN}+\beta_1]\lambda^{lN}y,\, lN\log(\lambda)\right)\,,\, xz\text{-plane}\right) \\
&= \log\left[\sqrt{1+\left(\frac{(\lambda^{lN}+\beta_1)y}{2}\right)^2} + \frac{(\lambda^{lN}+\beta_1)y}{2}\right] \\
&\geq \log\left[(\lambda^{lN}+\beta_1)y\right] \\
&= \log(y) + \log(1+\beta_1\lambda^{-lN}) + lN\log(\lambda) \\
&> lN\log(\lambda)
\end{aligned}$$

Here we are assuming that $\log(y) > 0$ or $y > 1$. We can ensure that both $x, y > 2$ (needed for a later application of Lemma 2.2) by replacing the loop at s corresponding to l_2 by a positive multiple of itself if necessary.

There is some point P_0 on $[A_1, A_2]$ which realizes $\operatorname{dist}_S(E_1, [A_1, A_2])$. From the first inequality we know that there is a point P_0' on the xz-plane such that $d_S(P_0, P_0') \leq \log(2)$. Then the triangle inequality gives

$$\begin{aligned}
\operatorname{dist}_S(E_1, [A_1, A_2]) &= d_S(E_1, P_0) \\
&\geq d_S(E_1, P_0') - d_S(P_0', P_0) \\
&\geq lN\log(\lambda) - \log(2)\,.
\end{aligned}$$

Thus $\operatorname{dist}_S(E_1, [A_1, A_2]) \geq \frac{lN}{2}\log(\lambda)$ provided $N \geq N_2 = \frac{2\log_\lambda(2)}{l}$.

Combining these estimates gives

$$\operatorname{dist}_S(E_1, \gamma_2) \geq C(N)$$

where $C(N) = \frac{lN}{2}\log(\lambda)$ provided $N \geq \max\{N_1, N_2\}$ as required. □

Step 2. (Convergence) The endpoints E_1 and F_1 of $[\gamma_1]_1$ and $[\gamma_2]_1$ lie on the Nl-level. Projecting them to the xy-plane via left multiplication by t^{-lN} gives points

$$t^{-lN}E_1 = \overline{l_2}^{\lambda^{lN}+\beta_1} = ([\lambda^{lN}+\beta_1]x,\, [\lambda^{lN}+\beta_1]y,\, 0)$$

and

$$\begin{aligned} t^{-lN}F_1 &= t^{-lN}\overline{l_2}\,\overline{l_1^N} \\ &= t^{-lN}(x,y,0)t^{lN} \\ &= (\lambda^{lN}x,\, \lambda^{-lN}y,\, 0)\,. \end{aligned}$$

These have difference vector

$$\langle \beta_1 x, D_1 y\rangle$$

where

$$D_1 = \lambda^{lN} + \beta_1 - \lambda^{-lN}\,. \tag{4.2}$$

Since left multiplication by t^{-lN} is an isometry we can estimate $d_S(E_1, F_1)$ in the xy-plane. Applying Lemma 2.2 (since $x, y > 2$) we obtain

$$\begin{aligned} d_S(E_1,F_1) &= d_S\left((0,0,0)\,,\,(\beta_1 x, D_1 y, 0)\right) \\ &\le 2\log\left[\frac{3\beta_1 x}{2}\right] \,+\, 2\log\left[\frac{3D_1 y}{2}\right]\,. \end{aligned}$$

Before giving a description of the convergence iteration, first look at the effect of right multiplication on difference vectors. Suppose $g_1, g_2 \in G$ lie on some k-level, and map isometrically (via left multiplication by t^{-k}) to points $(p_1, q_1, 0)$ and $(p_2, q_2, 0)$ in the xy-plane. They have difference vector

$$\langle d_1, d_2\rangle \;=\; \langle p_2 - p_1, q_2 - q_1\rangle\,.$$

Right multiplication by ${l_1}^m$ (where $m > 0$) gives points $g_1 t^{lm}$ on the $(k + lm)$-level, which project to points

$$\begin{aligned} t^{-lm}t^{-k}g_i t^{lm} &= t^{-lm}(p_i, q_i, 0)t^{lm} \\ &= (\lambda^{lm}p_i, \lambda^{-lm}q_i, 0)\,. \end{aligned}$$

The new difference vector is

$$\langle \lambda^{lm}d_1, \lambda^{-lm}d_2\rangle$$

which has slope $\lambda^{-2lm}\frac{y}{x}$.

If we only right multiply g_2 by $\overline{l_2}^n$ (for some $n > 0$) then the points g_1 and $g_2\overline{l_2}^n$ still lie on the k-level. They project to the points

$$t^{-k}g_1 \ = \ (p_1, q_1, 0)$$

and

$$\begin{aligned} t^{-k}g_2\overline{l_2}^n &= (p_2, q_2, 0)(nx, ny, 0) \\ &= (p_2 + nx, q_2 + ny, 0) \end{aligned}$$

in the xy-plane. These have difference vector

$$\langle d_1 + nx,\ d_2 + ny \rangle \, .$$

The convergence construction exploits these effects of right multiplication on difference vectors. Here is the precise description.

Convergence construction: Hypothesis. Assume that for some $j \geq 1$ the paths $[\gamma_1]_j$ and $[\gamma_2]_j$ have endpoints E_j and F_j which lie on the k_j-level. Suppose that these points project (via left multiplication) to the xy-plane with difference vector

$$\langle \beta_j x, D_j y \rangle$$

where β_j satisfies $\lambda^{2l} + 1 \leq \beta_j < \lambda^{2l} + 2$ and where $D_j \geq \lambda^{2l} + 2$. Recall that x and y are the coordinates of $\overline{l_2} = v$. Note that the slope of this vector is greater than $\frac{y}{x}$. In figure 3 (a) $t^{-k_j}E_j$ has smaller x- and y-coordinates than $t^{-k_j}F_j$.

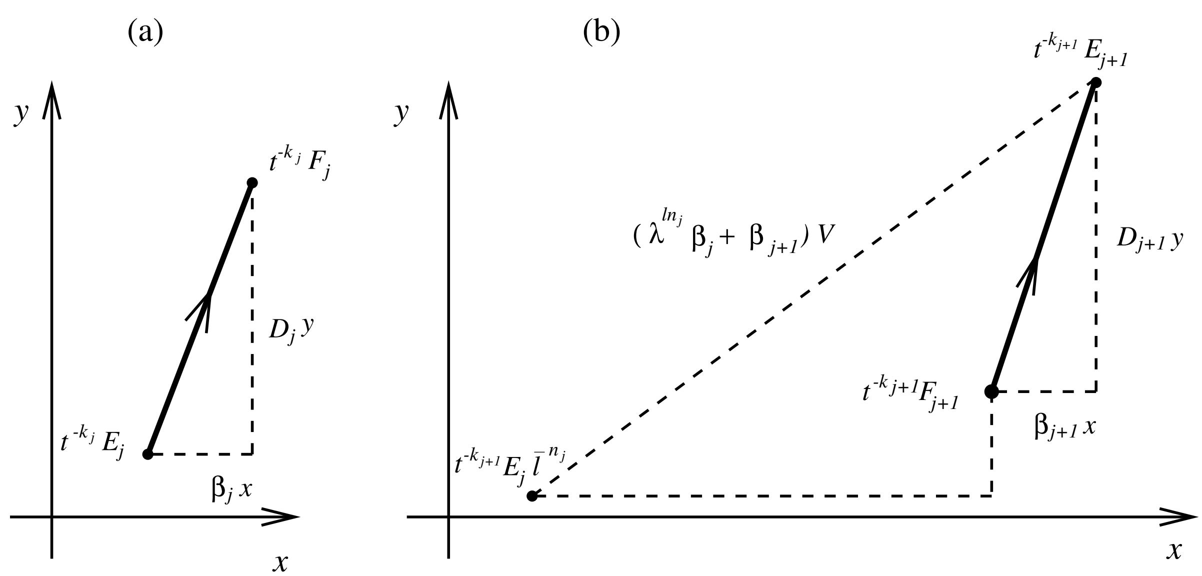

FIGURE 3. Difference vectors in the convergence construction.

Convergence construction: Inductive step. Let n_j be the greatest integer less than or equal to $\frac{1}{2}\log_{\lambda^l}(D_j)$ so that

$$D_j = \lambda^{l(2n_j+2\theta_j)} \quad \text{for some } 0 \le \theta_j < 1. \tag{4.3}$$

Since $D_j \ge \lambda^{2l}+2$ we see that n_j will be at least 1. Now define

$$[\gamma_1]_{j+1} \;=\; [\gamma_1]_j l_1^{n_j} l_2^{\lambda^{ln_j}\beta_j+\beta_{j+1}}$$

and

$$[\gamma_2]_{j+1} \;=\; [\gamma_2]_j l_1^{n_j}$$

where $\lambda^{2l}+1 \;\le\; \beta_{j+1} \;<\; \lambda^{2l}+2$ is chosen such that $\beta_{j+1}+\lambda^{ln_j}\beta_j \in \mathbb{N}$.

Let E_{j+1} and F_{j+1} be the endpoints of $[\gamma_1]_{j+1}$ and $[\gamma_2]_{j+1}$ respectively. Then they project (via left multiplication) to points in the xy-plane with difference vector

$$\langle \beta_{j+1}x, D_{j+1}y\rangle$$

where

$$\begin{aligned} D_{j+1} &= \lambda^{ln_j}\beta_j + \beta_{j+1} - \lambda^{-ln_j}D_j \\ &= \lambda^{ln_j}(\beta_j - \lambda^{2l\theta_j}) + \beta_{j+1} \end{aligned} \tag{4.4}$$

The second equality uses equation (4.3). Now the hypothesis on β_j implies that

$$\begin{aligned} D_{j+1} &\ge \lambda^{ln_j}(1) + \beta_{j+1} \\ &\ge \lambda^{ln_j}\,. \end{aligned}$$

For $n_j \ge N_3 = \frac{\log_\lambda(\lambda^{2l}+2)}{l}$ this becomes

$$D_{j+1} \ge \lambda^{2l}+2$$

and so we can iterate the procedure. The difference vector $\langle \beta_{j+1}x, D_{j+1}y\rangle$ is shown in figure 3 (b).

Note that the x- and y-coordinates of $t^{-(ln_j+k_j)}E_{j+1}$ will be larger than those of $t^{-(ln_j+k_j)}F_{j+1}$ this time, so the roles of the $[\gamma_i]_{j+1}$ will be reversed. After a few iterates the paths $[\gamma_i]_j$ will spiral around each other in Sol as shown in figure 4.

Convergence construction: Case $j=1$. We start the iteration at $j=1$. From equation (4.2) we know that the difference vector for the points $t^{-lN}F_1$ and $t^{-lN}E_1$ is $\langle \beta_1 x, D_1 y\rangle$ where

$$\begin{aligned} D_1 &= \lambda^{lN} + \beta_1 - \lambda^{-lN} \\ &= \lambda^{lN}(1+\beta_1\lambda^{-lN} - \lambda^{-2lN})\,. \end{aligned}$$

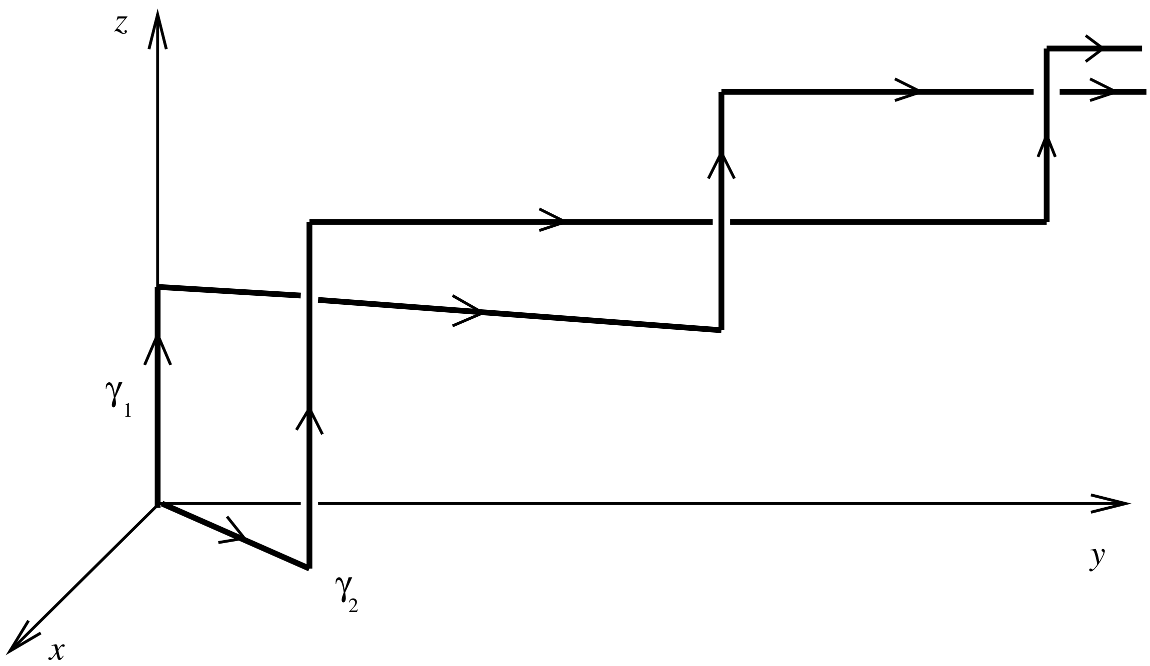

FIGURE 4. The paths γ_i in Sol

Now n_1 is the greatest integer less than or equal to $\frac{1}{2}\log_{\lambda^l}(D_1)$. Thus

$$n_1 = \frac{N}{2}$$

provided N is large enough so that $\log_{\lambda^l}(1+\beta_1\lambda^{-lN} - \lambda^{-2lN}) < 2$. This is true if

$$N \geq N_4 = \frac{1}{l}\log_\lambda\left[\frac{\lambda^{2l}+2}{\lambda^{2l}-1}\right].$$

Thus the path γ_2 will be of the form $[\gamma_2]_1 l_1^{\frac{N}{2}} u$ as described in Lemma 4.2, and so the divergence estimate holds.

We have defined the convergence construction inductively. The next lemma gives conditions on the n_j which ensure that the paths being constructed are actually converging.

Lemma 4.3. *If n_j is large enough in the convergence construction above, then*

$$d_S(E_{j+1}, F_{j+1}) \leq \frac{2}{3} d_S(E_j, F_j).$$

Exact lower bounds N_5 and N_6 for n_j are given in the proof.

Proof. We will work in the case where $t^{-k_j}E_j$ has smaller x- and y-coordinates than $t^{-k_j}F_j$. The same argument works in the other case.

Since left multiplication is an isometry we have

$$d_S(E_{j+1},F_{j+1}) \;=\; d_S(\,(0,0,0)\,,\,(\beta_{j+1}x, D_{j+1}y, 0)\,)\,.$$

Thus Lemma 2.2 and equation (4.4) give

$$\begin{aligned}
d_S(E_{j+1},F_{j+1}) &\le 2\log\left[\frac{3\beta_{j+1}x}{2}\right] \;+\; 2\log\left[\frac{3y}{2}D_{j+1}\right]\\
&= 2\log\left[\frac{3\beta_{j+1}x}{2}\right] \;+\; 2\log\left[\frac{3y}{2}\lambda^{ln_j}(\beta_j - \lambda^{2l\theta_j} + \lambda^{-ln_j}\beta_{j+1})\right]\\
&< 2\log\left[\frac{3}{2}(\lambda^{2l}+2)\right] \;+\; \log\left[\frac{3y}{2}\right] \;+\; 2\log(\lambda)ln_j\\
&+ 2\log\left[\beta_j - \lambda^{2l\theta_j} + \lambda^{-ln_j}\beta_{j+1}\right]\\
&< C_1 + 2\log(\lambda)ln_j
\end{aligned}$$

where

$$C_1 \;=\; 2\log\left[\frac{3}{2}(\lambda^{2l}+2)\right] \;+\; 2\log\left[\frac{3y}{2}\right] \;+\; 2\log\left[2(\lambda^{2l}+2)\right]\;.$$

Note that C_1 is a positive constant since $x, y > 1$.

On the other hand, Lemma 2.2 and equation (4.3) give

$$\begin{aligned}
d_S\,(\,(0,0,0)\,,\,(\beta_j x, D_j y, 0)\,) &\ge \max\left\{2\log\left[\beta_j x\right]\,,\;2\log\left[D_j y\right]\right\}\\
&\ge \max\left\{2\log\left[(\lambda^{2l}+1)x\right]\,,\;2\log\left[\lambda^{l(2n_j+2\theta_j)}y\right]\right\}\\
&\ge \max\left\{2\log\left[(\lambda^{2l}+1)x\right]\,,\;2(2ln_j)\log(\lambda)\right\}\\
&= 4ln_j\log(\lambda)
\end{aligned}$$

provided

$$n_j \;\ge\; N_5 \;=\; \frac{\log\left[(\lambda^{2l}+1)x\right]}{2l\log(\lambda)}\,.$$

Combining these estimates, we see that

$$d_S(E_{j+1},F_{j+1}) \;\le\; \frac{2}{3}d_S(E_j,F_j)$$

if

$$C_1 + 2ln_j\log(\lambda) \;\le\; \frac{8ln_j\log(\lambda)}{3}\,.$$

This is true if

$$n_j \;\geq\; N_6 \;=\; \frac{3C_1}{2l\log(\lambda)}\,.$$

The lemma follows, with N_5 and N_6 as the lower bounds for n_j. □

Here is a summary of what we have shown. We constructed $[\gamma_1]_1$ and $[\gamma_2]_1$, and found that the point E_1 on $[\gamma_1]_1$ lies outside a $C(N)$-neighborhood of all of γ_2, provided that

$$N \;\geq\; \max\{N_1, N_2, N_4\}$$

from Lemma 4.2.

We construct the paths $[\gamma_1]_j$ and $[\gamma_2]_j$ (with endpoints E_j and F_j respectively) by iteration on j, adjoining words from the monoid $\langle l_1, l_2\rangle$ to the endpoints of $[\gamma_i]_j$ to obtain $[\gamma_i]_{j+1}$. These paths have the property that

$$d_S(E_{j+1}, F_{j+1}) \;\leq\; \frac{2}{3} d_S(E_j, F_j)$$

provided

$$n_j \;\geq\; N_7 = \max\{N_3, N_5, N_6\}$$

from the construction in step 2.

Now, $n_1 = \frac{N}{2}$ (provided $N \geq N_4$) and so we must have

$$N \geq 2N_7$$

to start the convergence iteration. We continue the iteration until, for some $2 < r \in \mathbb{N}$, we have $n_r < N_7$. But then

$$\begin{aligned} d_S(E_r, F_r) &\leq 2\log\left[\frac{3x}{2}\beta_r\right] \;+\; 2\log\left[\frac{3y}{2}D_r\right] \\ &< 2\log\left[\frac{3x}{2}(\lambda^{2l}+2)\right] \;+\; 2\log\left[\frac{3y}{2}\lambda^{l(2n_r+2\theta_r)}\right] \\ &< 2\log\left[\frac{3x}{2}(\lambda^{2l}+2)\right] \;+\; 2\log\left[\frac{3y}{2}\lambda^{2l(N_7+1)}\right] \\ &= H \end{aligned}$$

and we just choose N large enough such that

$$H \;<\; \alpha_1 C(N) - \alpha_2.$$

Since $C(N) = \frac{Nl}{2}\log(\lambda)$ we see that

$$N \;\geq\; \frac{2(H+\alpha_2)}{\alpha_1 l \log(\lambda)}$$

works. Let $\gamma_i = [\gamma_i]_r$ and $C = C(N)$ where

$$N \gg \max\left\{N_1,\ N_2,\ N_4,\ 2N_7,\ \frac{2(H+\alpha_2)}{\alpha_1 l \log(\lambda)}\right\}.$$

Then γ_1 is not contained in a C-Hausdorff neighborhood of γ_2 in Sol, yet

$$d_S(\overline{\gamma_1}, \overline{\gamma_2}) \ \leq\ \alpha_1 C - \alpha_2$$

as in equation (4.1). This gives the required contradiction in the case where $l_1 = t^l$.

Remarks 4.4. In the general case $l_1 = t^l w$ where $w \in \langle a, b\rangle$ is not the identity. Suppose that w maps to the point $(x_1, y_1, 0)$ in Sol. Then $\frac{y_1}{x_1}$ will be irrational, so that x_1 and y_1 are nonzero. Now, for $n \in \mathbb{N}$ we have

$$\begin{aligned}(t^l w)^n &= t^{ln}\left(t^{-l(n-1)} w t^{l(n-1)}\right)\dots\left(t^{-l} w t^l\right) w \\ &= t^{ln}\left(\left[\frac{\lambda^{ln}-1}{\lambda^l - 1}\right] x_1\,,\ \left[\frac{\lambda^{-ln}-1}{\lambda^{-l}-1}\right] y_1\,,\ 0\right)\end{aligned} \tag{4.5}$$

Suppose that $g_1, g_2 \in G$ lie on some k-level, and that the points $g_i t^{ln}$ project (via left multiplication by $t^{-(k+ln)}$) to points P_i in the xy-plane. Right multiplying g_i by $(t^l w)^n$ and projecting to the xy-plane has the effect of adding the vector

$$\left(\left[\frac{\lambda^{ln}-1}{\lambda^l - 1}\right] x_1\,,\ \left[\frac{\lambda^{-ln}-1}{\lambda^{-l}-1}\right]\,,\ 0\right)$$

to P_i. In particular, the difference vector will be unchanged. Thus, replacing t^l by $t^l w$ has no effect on the estimates in the convergence construction above.

Equation (4.5) also shows that w has only a bounded effect on the divergence estimates in Lemma 4.2. Since $\lambda^{-l} < 1$ the terms

$$\left[\frac{\lambda^{-ln}-1}{\lambda^{-l}-1}\right]$$

are just the partial sums of a convergent geometric series, and so are bounded by

$$\frac{\lambda^l}{\lambda^l - 1}.$$

For example, suppose we want to estimate the distance from the point $v(t^l w)^n$ to the xz-plane, where $\frac{N}{2} \leq n \leq \frac{3N}{2}$. Since left multiplication by t^{-ln} is an isometry which preserves the xz-plane we have

$$\begin{aligned}&\mathrm{dist}_S(v(t^l w)^n\,,\ xz\text{-plane}) \\ &= \mathrm{dist}_S\left(\left(\lambda^{ln}x + \left[\frac{\lambda^{ln}-1}{\lambda^l-1}\right]x_1\,,\ \lambda^{-ln}y + \left[\frac{\lambda^{-ln}-1}{\lambda^{-l}-1}\right]y_1\,,\ 0\right)\,,\ xz\text{-plane}\right) \\ &= \log\left[\sqrt{1+\left(\frac{Y}{2}\right)^2} + \frac{Y}{2}\right]\end{aligned}$$

where $Y = y\lambda^{-ln} + \left[\frac{\lambda^{-ln}-1}{\lambda^{-l}-1}\right] |y_1|$. As $n \geq \frac{N}{2}$ this will be bounded for large N. This bound will replace the $\log(2)$ bound of Lemma 4.2. We can make similar modifications to the lower bound for the distance from the endpoint of $[\gamma_1]_1$ to the xz-plane. This gives the final contradiction in case I.

§5. Proof of Theorem 1: Case II

Let us recall the working assumption in case II: there is at most one type of loop at each live and accessible state of Δ. As in section 3, we view G as a subgroup of Sol and L-words as paths in Sol. We shall obtain a contradiction by exhibiting a word $w \in L$ and $R > 0$ such that the neighborhood property (3.3) holds

$$B_R(\overline{w}) \cap G \not\subset \overline{L}\,.$$

Here $B_R(\overline{w})$ is the radius R ball about $\overline{w}$ in Sol. The following properties of L-words will be important.

(1) If $w, w_1 \in L$ are such that

$$d_S(\overline{w}, \overline{w_1}) \;\leq\; R$$

then w_1 and w lie in K_1R-Hausdorff neighborhoods of each other, where

$$K_1 \;=\; \frac{\epsilon_2 K}{\epsilon_1}\,.$$

(2) If a and b are points on the L-word w, then the subsegment $[a, b]$ of w with endpoints a and b satisfies

$$[a, b] \;\subset\; B_{\delta_1 + K_1 d_S(a,b)}(a)$$

where

$$\delta_1 \;=\; \frac{(2K+1)q}{\epsilon_1}\,.$$

Note that property 1 is just the standard consequence of the asynchronous K-fellow traveler property of L, which has been translated into Sol using estimate (2.1). Similarly, property 2 is the Sol version of Lemma 1.2 on bounded backtracking of L-words.

Given a maximal stem in Δ one can add r loops to this stem, obtaining a standard maximal track T as in definition 3.6. Then every L-word that has T as its track will be of the form

$$w \;=\; u_1 l_1^{n_1} \ldots u_r l_r^{n_r} u_{r+1} \tag{5.1}$$

where $n_i \geq 1$ for all $1 \leq i \leq r$, and the sum of the lengths of the u_i is less than q. We say that w is obtained from T by *pumping each loop* at least M times if $n_i \geq M$ for all $1 \leq i \leq r$. The next lemma gives conditions which ensure that only maximal L-words can track standard maximal words.

Lemma 5.1. *Let T be a standard maximal L-track. Then, given any $R > 0$, there exists an integer $M \geq 1$ such that if we pump each loop of T at least M times we obtain L-words w with the following property. If $w_1 \in L$ is such that*

$$d_S(\overline{w}, \overline{w_1}) \leq R$$

then w_1 is a maximal L-word.

Proof. The word w is of the form described in equation (5.1) where $n_i \geq M$ for all $1 \leq i \leq r$. Thus, for $M \gg q$, the path w in Sol looks like the concatenation of r directed segments (the $l_i^{n_i}$), each of which either lies in some k-level or is 'transverse' to the k-levels, pointing 'upwards' or 'downwards' in Sol. These segments are connected together by 'hinges' (the u_i) of length strictly less than q, as shown in figure 5. Strictly speaking the segments l_i with zero t-exponent sum do not have to lie in a k-level. However, we are only looking at r words l_i, and each of these lies within $\frac{\text{length}(l_i)}{2}$ of their endpoints. Thus, the segments $l_i^{n_i}$ lie within a bounded neighborhood of the corresponding segments where we write the l_i in standard form

$$t^* a^* b^*.$$

So me may assume that the l_i are in standard form. For each $\leq i \leq r$, the path w can be viewed as consisting of three subsegments:

$$w_{i,0} \;=\; u_1 l_1^{n_1} \ldots u_i$$

and

$$l_i^{n_i}$$

and

$$w_{i,1} \;=\; u_{i+1} l_{i+1}^{n_{i+1}} \ldots u_{r+1}\,.$$

Denote the endpoints of $l_i^{n_i}$ by

$$P_i \;=\; \overline{w_{i,0}}$$

and

$$Q_i \;=\; \overline{w_{i,0} l_i^{n_i}}\,.$$

Now, if $w_1 \in L$ is such that $d_s(\overline{w}, \overline{w_1}) \leq R$, property 1 implies that w_1 lies in a K_1R-Hausdorff neighborhood of w. The first claim shows that w_1 cannot take any shortcuts in the K_1R-neighborhood of w.

Claim 1: Let w, $w_{i,0}$, and $w_{i,1}$ be as described above. If M is large (precise lower bounds M_1 and M_2 given in the proof) then for each $1 \leq i \leq r$ the K_1R-neighborhoods of $w_{i,0}$ and $w_{i,1}$ are disjoint.

Proof of claim 1. Suppose the neighborhoods have nonempty intersection. Then there are points x in Sol, a on $w_{i,0}$ and b on $w_{i,1}$ such that

$$d_S(a, x) \leq K_1R \quad \text{and} \quad d_S(b, x) \leq K_1R.$$

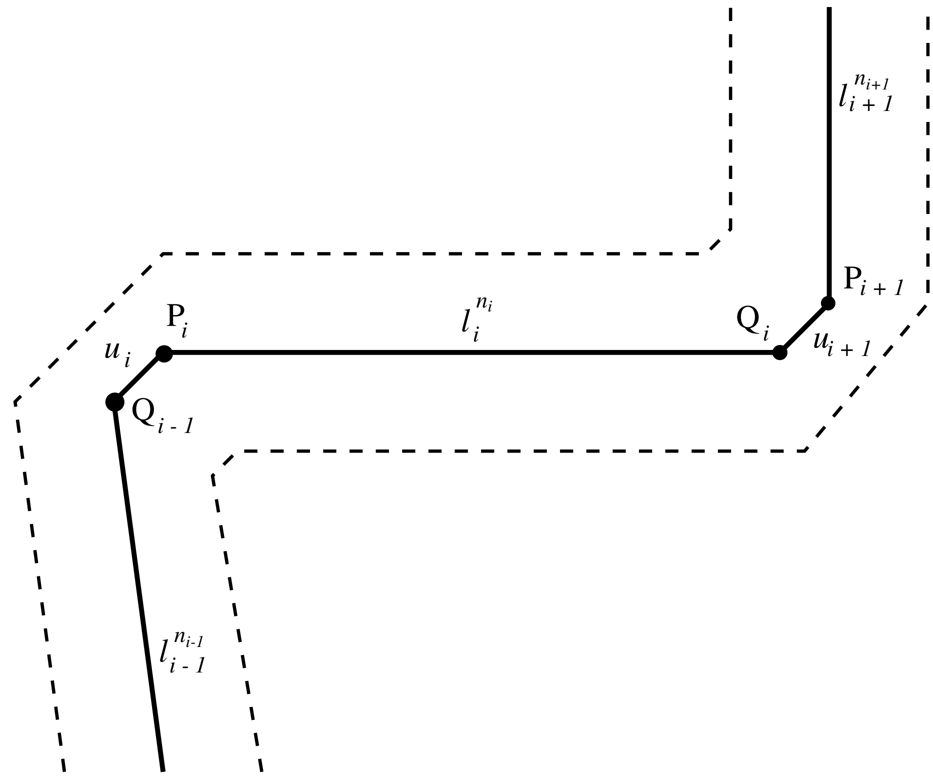

FIGURE 5. The Hausdorff neighborhood of w.

Then $d_S(a,b) \leq 2K_1R$ and property 2 implies that

$$[a,b] \subset B_{2K_1^2R+\delta_1}(a)$$

where $[a,b]$ denotes the subsegment of w with endpoints a and b. Now $l_i^{n_i}$ is a subsegment of $[a,b]$ so the distance between its endpoints is bounded by

$$2\delta_1 + 4K_1^2R\,.$$

This gives a contradiction for large values of M. Here are the precise bounds.

If l_i has nonzero t-exponent sum then

$$d_S(P_i,Q_i) \geq n_i \log(\lambda) \geq M\log(\lambda)$$

giving a contradiction if

$$M > M_1 = \frac{2\delta_1 + 4K_1^2R}{\log(\lambda)}\,.$$

If l_i has zero t-exponent sum then

$$\overline{l_i} = (x_i, y_i.0)\,.$$

Let

$$d = \min_j \{ |x_j| \, , \, |y_j| \}$$

where j runs over the elements of $\{i, \ldots, r\}$ for which $\overline{l_j} = (x_j, y_j, 0)$. Then

$$\begin{aligned} d_S(P_i, Q_i) &= d_S(\,(0,0,0)\,,\,(n_i x_i, n_i y_i, 0)\,) \\ &\geq d_S(\,(0,0,0)\,,\,(Md, Md, 0)\,) \\ &\geq 2 \log |Md| \end{aligned}$$

by Lemma 2.2. This gives a contradiction if

$$M \,>\, M_2 \,=\, \frac{1}{d} e^{2\delta_1 + 4K_1^2 R}$$

thus establishing claim 1. □

Now w_1 is a connected subset of the K_1R-neighborhood of w, with endpoints in the K_1R-neighborhoods of $w_{i,0}$ and $w_{i,1}$. Thus for each $1 \leq i \leq r$ there is at least one point c_i which lies in the K_1R-neighborhood of $l_i^{n_i}$, but not in the K_1R-neighborhoods of $w_{i,0}$ and $w_{i,1}$. The next claim states that w_1 tracks the segments $l_i^{n_i}$ almost up to their endpoints.

Claim 2: Let M be as in claim 1. Then for $1 \leq i \leq r$ there exist points $a_i, b_i \in w_1$ such that the subsegments $[a_i, b_i]$ of w_1 satisfy the following properties.

(1) $[a_i, b_i]$ is contained in the K_1R-Hausdorff neighborhood of $l_i^{n_i}$.
(2) $d_S(a_i, P_i) \leq N(R)$.
(3) $d_S(b_i, Q_i) \leq N(R)$.

where $N(R) \,=\, 2K_1^2 R + K_1 R + \delta_1$.

Proof of claim 2. For each $1 \leq i \leq r$ there exists $c_i \in w_1$ which lies in the K_1R-neighborhood of $l_i^{n_i}$ but not in the K_1R-neighborhoods of $w_{i,0}$ and $w_{i,1}$. Starting at c_i proceed along w_1 towards $\overline{w_1}$ until we reach a point b_i which also lies in the K_1R-neighborhood of $w_{i,1}$. Since b_i lies within K_1R of a point on $l_i^{n_i}$ and a point on $w_{i,1}$ these points are within $2K_1R$ of each other. Property 2 implies that the subsegment of w they define is contained in a $2K_1^2R + \delta_1$ radius ball about one of the points, and therefore is contained in a $2K_1^2R + K_1R + \delta_1$ ball about b_i. Thus

$$d_S(b_i, Q_i) \,\leq\, N(R) \,=\, 2K_1^2 R + K_1 R + \delta_1 \, .$$

Similarly, we can proceed back along w_1 towards the identity until we reach a point a_i which also lies in the K_1R-neighborhood of $w_{i,0}$. As above we have

$$d_S(a_i, P_i) \,\leq\, N(R) \, .$$

One may assume that a_i and b_i are vertices of the edgepath w_1 replacing $N(R)$ by $N(R)+\frac{1}{\epsilon_1}$ if necessary. We will also extend the last segment so that $b_r = \overline{w_1}$. This completes the proof of claim 2. □

We continue with the proof of Lemma 5.1. There are two cases to consider. The first is when l_i has nonzero t-exponent sum. Then one endpoint of $l_i^{n_i}$ is at least M levels above the other. Thus a_i is at least $M - \frac{2N(R)}{\log(\lambda)}$ levels above or below b_i. For

$$M \gg \frac{2N(R)}{\log(\lambda)} + q$$

these points are separated by more than q levels. Therefore the subsegment $[a_i, b_i]$ will traverse a nonzero type loop in the L-track corresponding to w_1 in Δ. Denote this track by T_1.

In the second case l_i has zero t-exponent sum. Now l_i must stay in some level, and the subsegment $[a_i, b_i]$ of w_1 cannot move more than $K_1 R$ above or below this level. It is possible for $[a_i, b_i]$ to remain inside the $K_1 R$ neighborhood of $l_i^{n_i}$ using alternating sequences of positive and negative t-exponent sum words. However, if M is large enough, this will contradict the fact that w is a maximal word. Here are the details.

A *simple* loop in Δ is a loop (as defined in section 2) which does not cross itself. Since Δ is finite, there are only finitely many simple loops. Thus there exists an integer $N_0 > 0$ such that the length of each simple loop of nonzero type is bounded by N_0 times the absolute value of the t-exponent sum of the loop. Now Case II hypothesis implies that any loop in Δ will be a combination of simple loops of the same type. Thus for every nonzero type loop l we have

$$\text{length}(l) \leq N_0 |t\text{-exponent sum of } l| .$$

Suppose that $[a_i, b_i]$ does not traverse a zero type loop in T_1. If a subsegment of $[a_i, b_i]$ traverses a positive loop its endpoints will be at least one level apart. Thus $[a_i, b_i]$ can traverse at most $\frac{2K_1R}{\log(\lambda)} + q$ positive loops before it has to 'change' to a negative loop. See figure 6. Therefore the lengths (in $\Gamma_A(G)$) of the subsegments of $[a_i, b_i]$ between loop changes are bounded by

$$N_0 \left(\frac{2K_1R}{\log(\lambda)} + q \right) + q$$

and estimate (2.3) implies that the Sol lengths are bounded by

$$B = \frac{N_0}{\epsilon_1} \left(\frac{2K_1R}{\log(\lambda)} + q \right) + \frac{q}{\epsilon_1} .$$

Now we can choose M large enough such that

$$\begin{aligned} d_S(a_i, b_i) &\geq 2\log(Md) - 2N(R) \\ &\gg 3B . \end{aligned}$$

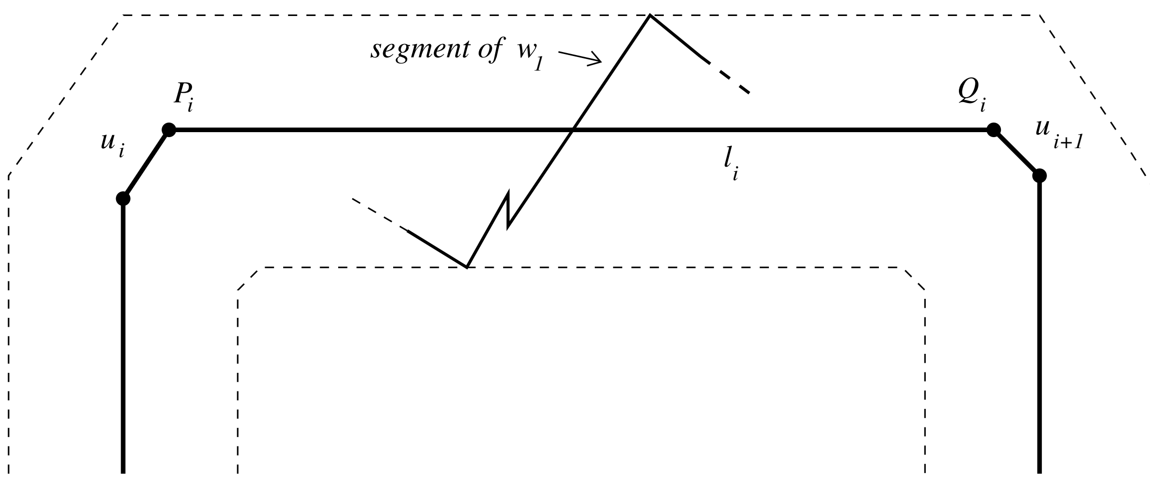

FIGURE 6. The K_1R-neighborhood of $l_i^{n_i}$.

and so T_1 will have more changes than T, contradicting the fact that T is a maximal track. Thus, $[a_i, b_i]$ must traverse zero type loops in T_1. We have seen that w_1 traverses positive, negative and zero type loops in Δ whenever w does, and so is a maximal L-word. This completes the proof of Lemma 5.1. $\square$

Finally, Lemma 5.2 below shows that L does not map onto G. This gives a contradiction in Case II and finally proves Theorem 1.

Lemma 5.2. *Given a standard maximal L-track T, there exists $R > 0$ and $w \in L$ whose L-track is T such that*

$$G \cap B_R(\overline{w}) \not\subset \overline{L}\,.$$

Proof. The proof breaks into two cases, depending on the type of the last loop along T. In the first case this loop is of zero type, and reading it gives a word l_r where

$$\overline{l_r} = (x_r, y_r, 0)\,.$$

There are only finitely many simple loops of zero type in Δ. Each of these gives a word which maps to the xy-plane in Sol as a vector with irrational slope. Choose constants $0 < A < B$ such that the absolute values of these slopes are bounded between A and B. Note that there is at most one type of loop at each live and accessible state of Δ, and thus each such loop is simple. This is because two or more such loops would give rise to a sublanguage of L with exponential growth which maps into some coset of $\langle a, b\rangle$ in G. But $\langle a, b\rangle \cong \mathbb{Z}^2$ has polynomial growth, and so two or more words in this language would have to map to the same point in the coset, contradicting the fact that L is a language with uniqueness.

Choose $R_0 \in \mathbb{N}$ such that

$$R_0 \gg 2q + \frac{\log_\lambda \left[\frac{B}{A}\right]}{2}$$

and set

$$R = R_0 \log(\lambda) .$$

Given R as above we pump each loop of T at least M times (where M is given in Lemma 5.1) to obtain the word w. The last loop of T is pumped sufficiently large so that we get a contradiction as follows.

The point $\overline{w}\bar{t}^{R_0}$ lies in the R ball about $\overline{w}$. By Lemma 5.1 if $w_1 \in L$ is such that

$$\overline{w_1} = \overline{w}\bar{t}^{R_0}$$

then w_1 is also a maximal L-word. Furthermore, the subsegment $[a_r, b_r]$ of w_1 is of the form

$$\ldots v_{s-1}^{m_{s-1}} v'_s v_s^{m_s} v'_{s+1}$$

where $m_i \in \mathbb{N}$, the v_i all have zero t-exponent sum, and the sum of the lengths of the v'_i is less than q. Actually, the beginning of $[a_r, b_r]$ may be a path which traverses a nonzero loop in Δ, but it can only do this for a bounded amount.

Translate w and w_1 by an isometry (left multiplication) so that the initial point P_r of the subsegment $l_r^{n_r}$ becomes the origin. Then $l_r^{n_r}$ is the vector $(n_r x_r, n_r y_r, 0)$ in the xy-plane, which has slope less than or equal to B. The $v_i^{m_i}$ lie somewhere between the $(R_0 - 2q)$ and $(R_0 + 2q)$ levels. Right multiplication by appropriate powers of t gives points X_{i_j} on the xy-plane which are within $(R_0 + 2q)\log(\lambda)$ of the vertex points of the subsegment $v_i^{m_i}$ of w_1. Thus these points lie in a $[K_1 R + (R_0 + 2q)\log(\lambda)]$-neighborhood of $l_r^{n_r}$. Lemma 2.2 implies that they lie in a bounded Euclidean neighborhood of $l_r^{n_r}$. We know from section 4 the effect of right multiplication on difference vectors. The points X_{i_j} will have difference vectors with slope greater than

$$\lambda^{2(R_0 - 2q)} A \gg \lambda^{\log_\lambda\left(\frac{B}{A}\right)} A = B .$$

But $l_r^{n_r} = (n_r x_r, n_r y_r, 0)$ has slope bounded above by B, and so we get a contradiction for n_r sufficiently large.

In the second case the last loop along T is of nonzero type. We suppose that it is a positive loop, and that its corresponding word l_r has t-exponent sum equal to $m > 0$. There are only finitely many simple positive loops in Δ. They are of the form $t^{p_j} u'_j$ where $p_j > 0$ and

$$u'_j = (x_j, y_j, 0)$$

with j in some finite indexing set J. Let

$$Y = \max_{j \in J} |y_j| .$$

The hypothesis of Case II implies that each positive loop at a live and accessible state of Δ must be a combination of the simple loops above.

We choose $R = R(Y)$ sufficiently large (lower bound estimated below), and use Lemma 2.2 to obtain a corresponding $M > 0$. As in the first case, $w \in L$ is chosen by pumping each loop of T at least M times, and pumping the last loop enough to obtain the contradiction below.

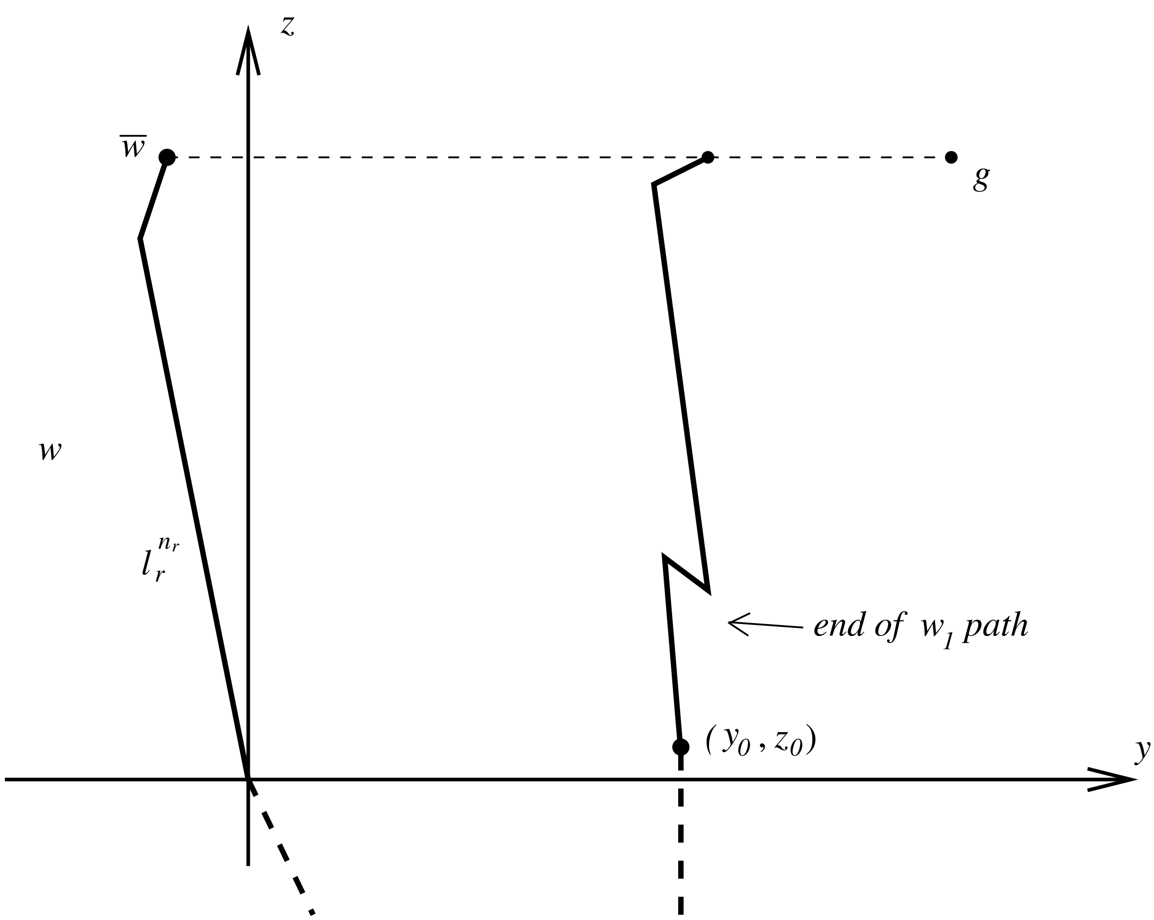

FIGURE 7. The image of L-paths in the yz-plane.

Pick a point on the line containing $\overline{w}$ and parallel to the y-axis, which is a Sol distance approximately R from $\overline{w}$. Now choose $g \in G \subset \text{Sol}$ which is nearest this point, so that

$$g \in G \cap B_R(\overline{w}) .$$

Figure 7 shows the points $\overline{w}$ and g projected onto the yz-plane by the map $(x, y, z) \mapsto (y, z)$. Translate the path w isometrically (by left multiplication) so that the initial point P_r of

$l_r^{n_r}$ is the origin. If $w_1 \in L$ is such that $\overline{w_1} = g$ then Lemma 5.1 implies that w_1 is a maximal L-word. Furthermore, its final section (all but a bounded amount of $[a_r, b_r]$) is of the form

$$\ldots v_{s-1}^{m_{s-1}} v_s' v_s^{m_s} v_{s+1}'$$

where the v_i are now positive t-exponent sum words, and the sum of the lengths of the v_i' is less than q. Pick a point on the segment $[a_r, b_r]$; for example the initial point of the first of the v_i. It maps to a point (y_0, z_0) in the yz -plane.

Recall that l_r is a word with $m > 0$ t-exponent sum, and is a product of words of the form

$$t^{p_j} u_j'$$

described above. Therefore, the distance from the endpoint of $\overline{l_r}^{n_r}$ to the xz-plane is bounded above by

$$\begin{aligned}\mathrm{dist}_S\left([t(0,Y,0)]^{mn_r} , xz\text{-plane} \right) &= \mathrm{dist}_S\left(\left(0\,,\ \frac{\lambda^{-mn_r} - 1}{\lambda^{-1} - 1} Y\,,\ 0\right) ,\ xz\text{-plane} \right) \\ &\leq \mathrm{dist}_S\left(\left(0\,,\ \frac{\lambda Y}{2(\lambda - 1)}\,,\ 0\right) ,\ xz\text{-plane} \right)\end{aligned}$$

for n_r large enough. By Lemma 2.3 the distance from $\overline{w}$ to the xz-plane is bounded by

$$\log\left[\sqrt{1 + \left(\frac{\lambda Y}{2(\lambda - 1)}\right)^2} + \frac{\lambda Y}{2(\lambda - 1)}\right] + \frac{q}{\epsilon_1}\,. \tag{5.2}$$

Now the point (y_0, z_0) lies on some k-level (where $z_0 = k\log(\lambda)$) and so

$$(0, y_0, z_0) \ = \ (0, y_0, 0)t^k\,.$$

Since the v_j are products of words of the form $t^{p_j} u_j'$ as well, the distance from the endpoint of

$$\ldots v_{s-1}^{m_{s-1}} v_s' v_s^{m_s} v_{s+1}'$$

to the xz-plane is bounded above by

$$\begin{aligned}&\mathrm{dist}_S\left((0, y_0, z_0)[t(0,Y,0)]^{mn_r - k} , xz\text{-plane} \right) + \frac{q}{\epsilon_1} \\ &= \mathrm{dist}_S\left(\left(0\,,\ \lambda^{-mn_r} y_0 + \frac{\lambda^{-(mn_r - k)} - 1}{\lambda^{-1} - 1} Y\,,\ 0\right) ,\ xz\text{-plane} \right) + \frac{q}{\epsilon_1} \\ &\leq \mathrm{dist}_S\left(\left(0\,,\ 1 + \frac{\lambda Y}{2(\lambda - 1)}\,,\ 0\right) ,\ xz\text{-plane} \right) + \frac{q}{\epsilon_1}\end{aligned}$$

for n_r large enough. Lemma 2.3 implies that the distance from this endpoint (which is $\overline{w_1}$) to the xz-plane is bounded by

$$\log\left[\frac{3}{2}\left(1+\frac{\lambda Y}{2(\lambda-1)}\right)\right] + \frac{q}{\epsilon_1}. \tag{5.3}$$

Thus the distance between $\overline{w}$ and $\overline{w_1}$ is less than the sum of the bounds in thetag5.2 and thetag5.3. If $R = R(Y)$ is chosen much larger than this sum to begin with (note that $R(Y)$ is independent of M and n_r), then we would have $d_S(\overline{w},\overline{w_1}) = d_S(\overline{w},g)$ is approximately R, a contradiction. This establishes Lemma 5.2 and so completes the proof of Theorem 1. □

§6. Remarks and questions

Since Sol is a subgroup of index eight in its full isometry group, a result of [NS] shows that Theorem 1 extends to include lattices in Isom(Sol).

Sol geometry groups also appear as subgroups of the Hilbert Modular groups, so the following is a natural question to ask.

Question 6.1. Are the Hilbert Modular groups asynchronously automatic?

Consider, for example, the group $G = \mathrm{Sl}_2(\mathbb{Z}(\sqrt{2}))$. The Borel subgroup $B < G$ (consisting of all upper triangular matrices) is a Sol group. We can see this as follows. The units in the ring $\mathbb{Z}(\sqrt{2})$ form an infinite cyclic group generated by the element $\epsilon = \sqrt{2}+1$. There is a corresponding infinite cyclic subgroup $\mathbb{Z} < B$ generated by the diagonal matrix with entries ϵ and $\epsilon^{-1} = \sqrt{2}-1$. The subgroup $N < B$, consisting of matrices with 1 along the diagonal and entries $a+b\sqrt{2}$ $(a,b\in\mathbb{Z})$ above the diagonal, is a normal subgroup which is isomorphic to $\mathbb{Z}^2$. The group B is a semi direct product $N \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on N by conjugation. Thus B is isomorphic to the Sol group $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$, where

$$\phi = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

Unfortunately, knowing that a subgroup of a group G is not asynchronously automatic, does not necessarily tell us about the group itself. Perhaps some of the techniques used in the Sol case can be applied here too. The group G is an irreducible, (non-cocompact) lattice in $\mathrm{Isom}(\mathbb{H}^2\times\mathbb{H}^2)$.

Thurston has asked if the cocompact irreducible lattices in $\mathrm{Isom}(\mathbb{H}^2\times\mathbb{H}^2)$ are automatic. Ones attempts at Question 6.1 might yield insights into this intriguing question.

References

[AB] J. Alonso and M. Bridson, *Semihyperbolic groups*, Proc. London Math. Soc. **70** **1** (1995), 56–114.

[BGSS] G. Baumslag, S.M. Gersten, M. Shapiro, and H. Short, *Automatic groups and amalgams*, Jour of Pure and Applied Algebra **76** (1991), 229–316.

[Bra] N. Brady, *Asynchronous automatic structures on closed hyperbolic surface groups*, Geometric Group Theory (R. Charney, M. Davis, M. Shapiro, eds.), de Gruyter, Berlin, New York, 1995, pp. 49–65.

[BBr] N. Brady and M. Bridson, *On graphs of abelian groups and the absence of biautomaticity*, Preprint (1999).

[Bri] M. Bridson, *Combings of semidirect products and* 3*-manifold groups*, Geom. Funct. Anal. **3** (1993), 263–278.

[BG] M. Bridson and R. Gilman, *On bounded languages and the geometry of nilpotent groups*, Combinatorial and geometric group theory – Edinburgh, London Math. Soc. Lecture Note Ser., vol. 204, Cambridge Univ. Press, 1995, pp. 1–15.

[C] J.W. Cannon, *The combinatorial structure of cocompact discrete hyperbolic groups*, geometricae Dedicata **16** (1984), 123–148.

[ECHLPT] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Patterson, W.P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publ. (1992).

[G] M. Grayson, *Geometry and growth in 3-dimensions*, PhD Thesis. Princeton University (1983).

[NS] W. Neumann and M. Shapiro, *Equivalent automatic structures and their boundaries*, International Journal of Algebra and Computation **2** (1992), 443–469.

[Sc] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.

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