Finite Subgroups of Hyperbolic Groups

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Abstract

We prove that every finite subgroup of a hyperbolic group G can be conjugated to a $2\delta+1$ neighborhood of the identity element, where δ is the hyperbolicity constant for G with respect to a given generating set. This gives an upper bound for the size of such finite subgroups in terms of δ and the number of generators for G.

A major theme in geometric group theory is the idea that one may better understand a group by having it act properly discontinuously by isometries on a suitable metric space or *geometry*. One can then hope to use geometric properties of the space to conclude facts about the group. [1]

For instance, if the geometry is negatively curved (for example hyperbolic space) then bounded subsets have unique centers. A group of isometries which fixes a bounded subset must also fix its center, and hence has a fixed point. A classical example of this is E. Cartan's theorem which states that a compact subgroup of the group of isometries of a simply connected riemannian manifold M of negative curvature fixes a point of M. This result is used in [8] to count conjugacy classes of finite subgroups of finitely generated Kleinian groups. Similar fixed point results are proven in [6], and are used to analyze finite subgroups of groups acting on Euclidean buildings—see also [5] page 157.

The purpose of this note is to show that these fixed point ideas carry over to the case of a hyperbolic group G acting by left translation on its Cayley graph. One can define a center of a bounded subset S of the Cayley graph. Since we only have a coarse version of hyperbolicity in this case, centers may not be unique. Nevertheless, we show that the coarse hyperbolicity (termed δ-hyperbolicity) ensures that the diameter of the set of centers of S is bounded independently of S. We use this to obtain a coarse version of a

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fixed point theorem: namely, that each finite subgroup of G permutes a set of points of small diameter. Hence, the finite subgroups can be conjugated into a small neighborhood of the identity. This argument is from the authors PhD thesis — [3]. It has been independently discovered by other researchers in the field — see the remark following the proof of Theorem 1.31 on page 140 of [4]. An alternative proof of the finiteness of the number of conjugacy classes of finite subgroups can be found in [2].

In Section 1 we give definitions of δ-hyperbolic metric spaces and hyperbolic groups, and state the main result—Theorem 1.1. In Section 2 we examine the notion of *center* of a bounded subset of a δ-hyperbolic metric space, and give a proof of Theorem 1.1.

1 Definitions

Let G be a finitely generated group with finite generating set $A \subset G$ which is closed with respect to inversion. We define the *Cayley graph* of G with respect to the generating set A (denoted by $\Gamma_A(G)$) as follows. The vertex set of $\Gamma_A(G)$ is the group G, and for each $a \in A$ and each $g \in G$ there is an edge from the vertex g to the vertex ga. We make $\Gamma_A(G)$ into a metric space by requiring that each edge be isometric to the unit interval, and taking the induced path metric.

A path connected metric space X with distance function d is called a *geodesic* metric space if for all points x, y in X there is an isometric map from the interval $[0, d(x,y)]$ to a path in X joining x to y. For example, if G is a finitely generated group then the Cayley graph of G with the usual path metric is a geodesic metric space.

Next we impose a combinatorial version of negative curvature on the Cayley graph of a finitely generated group. This is achieved by requiring that geodesic triangles in the Cayley graph are uniformly thin. These ideas first appeared in [10]. We work with the definitions given in [12].

Given three points x, y, z in a geodesic metric space X, choose geodesics joining them to form a triangle Δxyz. By the triangle inequality there is a comparison triangle $\Delta x'y'z'$ in the Euclidean plane. This means $d(x,y) = d_E(x',y')$, $d(x,z) = d_E(x',z')$ and $d(y,z) = d_E(y',z')$ where d_E denotes the Euclidean metric. The maximum inscribed circle meets $\Delta x'y'z'$ in points a, b, c as shown in Figure 1, where $d_E(x',b) = d_E(x',c)$, $d_E(y',a) = d_E(y',c)$, and $d_E(z',a) = d_E(z',b)$. Collapse this comparison triangle down to a tripod T by isometrically identifying the segment $[x', b]$ with $[x', c]$, $[y', a]$ with $[y', c]$, and $[z', a]$ with $[z', b]$. The resulting composed map $f : \Delta xyz \to T$ is an isometry on the edges of Δxyz. We say that the geodesic triangle Δxyz is δ*-thin* if the diameter of $f^{-1}(p)$ is bounded above by δ for all $p \in T$.

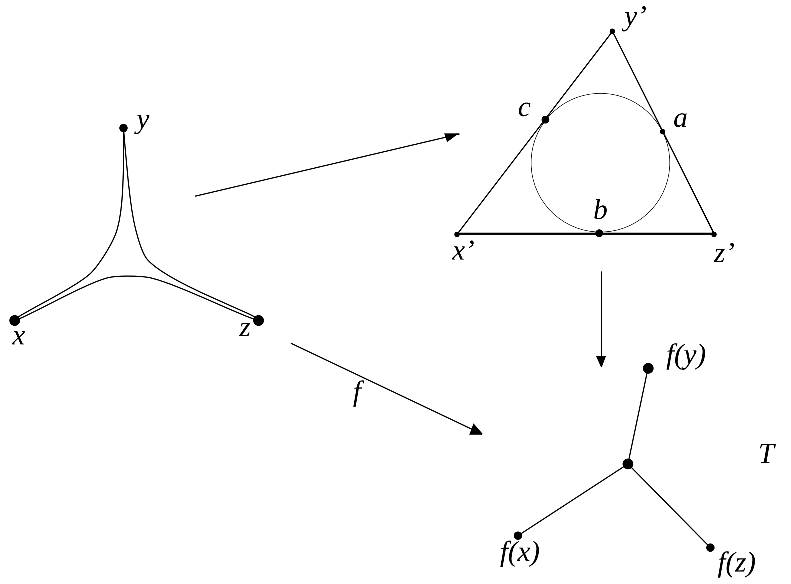

Figure 1: δ-thin triangles

A path connected geodesic metric space X is called δ*-hyperbolic* if there exists $\delta \geq 0$ such that every geodesic triangle in X is δ-thin. A finitely generated group G is called *hyperbolic* if the Cayley graph $\Gamma_A(G)$ is a δ-hyperbolic metric space. In [12] it is shown that if B is another finite generating set then $\Gamma_B(G)$ is δ'-hyperbolic for some $\delta' \geq 0$, so the property of being a hyperbolic group is independent of the finite generating set.

We are now in a position to state the main theorem.

Theorem 1.1 *Let G be a hyperbolic group, which is a δ-hyperbolic metric space in the word metric with respect to the generating set A. Then every finite subgroup of G can be conjugated to lie within the $(2\delta + 1)$-ball about the identity element.*

The Cayley graph $\Gamma_A(G)$ is a locally finite graph, with each vertex having valence $|A|$. Hence the following Corollary is immediate.

Corollary 1.2 *Let G be a hyperbolic group as above. Then there are only finitely many conjugacy classes of finite subgroups of G. Every finite subgroup contains at most $|A|^{2\delta+1}+1$ elements.*

2 Finite subgroups of hyperbolic groups

We begin by defining the notion of a center of a bounded subset of a metric space. This idea appeared in [6] where it was used in proving a fixed point theorem for finite groups of isometries of Euclidean buildings. Let X be a proper metric space (i.e. one for which closed metric balls are compact). Consider a set $S \subset X$ of bounded diameter. To each point x in X we associate a number $r_x \in \mathbf{R}_+$ by

$$r_x \;=\; \inf\{\, r > 0 \;:\; S \subset B_r(x)\,\}$$

where $B_r(x)$ denotes the ball of radius r about x in X. Since S is bounded this number is well defined, and if S contains two or more points r_x will be positive. We say that x_0 is a *center* of S if

$$r_{x_0} \;=\; \inf_{x\in X}\{r_x\}.$$

Note that any x_0 which minimize r_x must lie in a ball of radius equal to the diameter of S about any point of S. This ball is compact, since X is proper, and so such x_0 do exist. Note that $\Gamma_A(G)$ above is a proper metric space since it is a locally finite graph. Thus centers exist in the Cayley graphs of finitely generated groups.

A set may have more than one center. For example, consider a graph Γ which is the boundary of a pentagon. We define a metric on Γ by requiring that each edge is isometric to the unit interval, and taking the induced path metric. Let S be the set of vertices of Γ. It is easy to see that $r_0 = 2$ and that the set of centers of S is just S itself.

Now consider the metric space obtained from Γ by attaching a copy of the half line $[0,\infty)$ to each vertex (identifying 0 with the vertex). The resulting space (with the path metric) is a 2.5-hyperbolic metric space. Let S' be the set of points which are distance 100 along each of these half lines. Then the set of centers of S' is still the set of five vertices of Γ. The diameter of the set of centers is small, even though S' itself has large diameter. This phenomenon is a consequence of hyperbolic geometry as the next lemma shows.

Lemma 2.1 (set of centers is uniformly bounded) *Let X be a proper δ-hyperbolic metric space, and $S \subset X$ have bounded diameter. Then the set of centers of S is nonempty and has diameter bounded above by 2δ.*

Proof. We have seen above that, since X is proper, S will have at least one center.

Suppose x_1 and x_2 are two centers of S, and let

$$R = \inf\{ r > 0 : S \subset B_r(x_i) \}.$$

Join x_1 and x_2 by a geodesic, and let m be the midpoint. For every $\epsilon > 0$ there is an $s \in S$ such that $d(m, s) > R - \epsilon$. Draw in geodesics from x_i to s to get a geodesic triangle. Let f denote the usual map from the geodesic triangle $\Delta x_1 x_2 s$ to a tripod T, which is an isometry on the edges. Suppose that $a \in [x_2, s]$, $b \in [x_1, s]$, and $c \in [x_1, x_2]$ all map to the same point of T. We may assume without loss of generality that $d(c, x_2) \leq d(m, x_2)$ as shown in Figure 2.

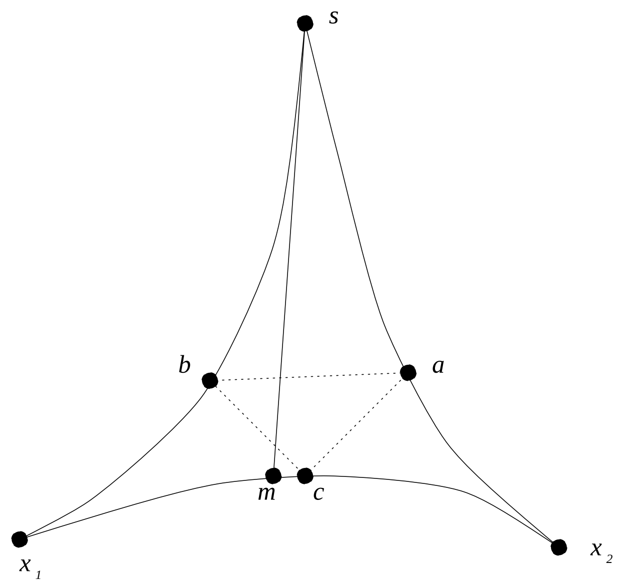

Figure 2: Estimating the diameter of a set of centers

Since $[x_1, x_2]$ is a geodesic, we have

$$d(m, c) + d(x_1, m) = d(x_1, c)$$

and

$$d(x_2, m) - d(m, c) = d(x_2, c).$$

Thus

$$\begin{aligned} R - \epsilon &< d(m, s) \\ &\leq d(m, c) + d(c, b) + d(b, s) \\ &\leq \frac{1}{2}[d(x_1, c) - d(x_2, c)] + \delta + d(b, s). \end{aligned}$$

Now since $[x_1, s]$ is a geodesic and x_1 is a center for S, we have

$$d(x_1, b) + d(b, s) = d(x_1, s) \leq R.$$

Combining this with the previous inequality gives

$$d(x_1, b) + d(b, s) \leq \frac{1}{2}[d(x_1, c) - d(x_2, c)] + \delta + \epsilon + d(b, s).$$

But $d(x_1, b) = d(x_1, c)$ and so this becomes

$$\frac{1}{2}[d(x_1, c) + d(x_2, c)] \leq \delta + \epsilon.$$

Thus

$$d(x_1, x_2) = d(x_1, c) + d(x_2, c) \leq 2\delta + 2\epsilon.$$

But this is true for any $\epsilon > 0$ and so $d(x_1, x_2) \leq 2\delta$ as required. $\square$

This a priori bound on the diameter of every set of centers gives us a coarse fixed point theorem for finite subgroups of hyperbolic groups. This gives a conceptually simple proof of the main result.

Proof of Theorem 1.1. Let $F < G$ be a finite subgroup of the hyperbolic group G. Let $A \subset G$ be a finite generating set, and suppose that $\Gamma_A(G)$ is a δ-hyperbolic metric space.

We can view F as a finite set of vertices in $\Gamma_A(G)$. As such, it is a bounded subset of $\Gamma_A(G)$, and so has a center. Furthermore, Lemma 2.1 implies that the set Cent (F) of centers of F has diameter at most 2δ. Now F is a set of vertices, and so Cent (F) will consist of points which are either vertices or midpoints of edges. Thus we can pick a vertex g_0 satisfying

$$d(g_0, \mathrm{Cent}(F)) \leq \frac{1}{2}$$

where $d(\ , \ )$ denotes the distance from a point to a set. Then, since left multiplication by g_0^{-1} is an isometry, ${g_0}^{-1}(\mathrm{Cent}(F))$ lies in a ball of radius $2\delta + \frac{1}{2}$ about the identity.

Note that if f is an isometry of $\Gamma_A(G)$ and x is a center of F, then $f(x)$ is a center of $f(F)$. This is because the definition of center only involves the metric, and f is an isometry. Now left multiplication by elements of F are isometries which leave the set F invariant. Hence they leave Cent (F) invariant as well.

Elements of the conjugate group ${g_0}^{-1}Fg_0$ leave the sets ${g_0}^{-1}(F)$ and ${g_0}^{-1}(\mathrm{Cent}(F))$ invariant. For any $h \in {g_0}^{-1}Fg_0$ and $x \in {g_0}^{-1}(\mathrm{Cent}(F))$ we have $h(x) \in {g_0}^{-1}(\mathrm{Cent}(F))$, and thus $d(x, h(x)) \leq 2\delta$. It is clear from the choice of g_0 that one can choose $x \in {g_0}^{-1}(\mathrm{Cent}(F))$ such that $d(x, e) \leq \frac{1}{2}$, where $e \in G$ is the identity element. Therefore,

$$\begin{aligned} d(e,h) &= d(e, h(e)) \\ &\leq d(e,x) + d(x,h(x)) + d(h(x),h(e)) \\ &\leq \frac{1}{2} + 2\delta + \frac{1}{2} \\ &= 2\delta + 1. \end{aligned}$$

That is ${g_0}^{-1}Fg_0 \subset B_{2\delta+1}(e)$, or in words: we can always conjugate a finite subgroup of G into a $(2\delta + 1)$-neighborhood of the identity. $\square$

Remarks. It was already known that there are only finitely many conjugacy classes of finite order elements in hyperbolic groups—see [9]. The proof involved homological arguments, and the action of G on a locally finite, finite dimensional, contractible complex called the Rips complex of G. In [11] Wing Poon used this result to prove that there are only finitely many conjugacy classes of finite solvable subgroups in G. His bounds involved the length of the composition series for a finite solvable subgroup.

By varying the generating set, we can get many Cayley graphs with different hyperbolic constants, δ. It is worth looking for an optimal generating set which will give a minimal value for δ, and hence a sharper bound in Corollary 1.2 for the order of finite subgroups.

We would like to be able to extend this result to larger classes of groups; for example, automatic groups [7], or semihyperbolic groups [1]. However, we cannot find an analog of lemma 2.1 (on bounding the diameter of the set of centers) for these groups. Another direction is to look at groups which act on $\mathbf{H}^n$ as a discrete group of isometries with finite volume quotient. There are stronger results in dimension 3: in [8] they show that a finitely generated, discrete subgroup of $\mathrm{Isom}(\mathbf{H}^3)$ has only finitely many conjugacy classes of finite subgroups. Their methods are topological, and do not give bounds on the number of conjugacy classes of finite subgroups in terms of the number of generators.

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