LOCAL-TO-ASYMPTOTIC TOPOLOGY FOR COCOMPACT CAT(0) COMPLEXES

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ABSTRACT. We give a local condition that implies connectivity at infinity properties for CAT(0) polyhedral complexes of constant curvature. We show by various examples that asymptotic-to-local results will be difficult to achieve. Nevertheless, we are able to prove a partial converse to our main local-to-asymptotic result.

1. INTRODUCTION

Given a finite complex X, it would be useful to have a local condition that implies that the universal cover $\widetilde{X}$ is n-acyclic or n-connected at infinity. For example, in [4], simple link conditions are given that ensure that a simply connected, CAT(0) cubical complex has prescribed topological properties at infinity. Namely,

Theorem 1.1 (4.1 in [4]). *Let X be a finite, locally* CAT(0) *cubical complex. If the link $Lk(v)$ of each vertex v is n-acyclic, and the link remains n-acyclic whenever you remove a simplex from $Lk(v)$, then the universal cover $\widetilde{X}$ is n-acyclic at infinity. Similarly, if $Lk(v)$ is simply connected and the link remains simply connected whenever you remove a simplex from $Lk(v)$, then the universal cover $\widetilde{X}$ is simply connected at infinity.*

The goal in this paper is to generalize Theorem 1.1 to arbitrary finite, locally CAT(0) complexes. Although the statement of Theorem 1.1 does not carry over to the general case, the basic argument of [4] does extend. In the general setting the key definition is that of a punctured link.

Definition 1.2 (Punctured link). Let X be a locally CAT(0) complex, let σ be a cell in X, let p be a point in $\mathrm{Lk}(\sigma)$, and recall that the "angular metric" induces a natural CAT(1) metric on $\mathrm{Lk}(\sigma)$. The *punctured link of σ at p* is obtained by removing from $\mathrm{Lk}(\sigma)$ all points within $\pi/2$ of p. The punctured link will be denoted $\mathrm{PLk}(\sigma, p)$. Notice that even though a punctured link of σ is not in general a subcomplex of $\mathrm{Lk}(\sigma)$, it nevertheles deformation retracts onto the maximal subcomplex of $\mathrm{Lk}(\sigma)$ that is contained in $\mathrm{PLk}(\sigma, p)$. Moreover, if X is a cubical complex, then $\mathrm{Lk}(\sigma)$ minus

[1]Supported under NSF grant no. DMS-0124344
[2]Supported under NSF grant no. DMS-0101506
Date: October 6, 2001.

the simplex containing p in its interior will deformation retract to the exact same maximal subcomplex.

The following is our main result.

Theorem 1.3. *Let X be a finite, locally-*$\mathrm{CAT}(0)$*, piecewise-Euclidean complex. If for each cell σ in X and for each $p \in Lk(\sigma)$, the spaces $Lk(\sigma)$ and $PLk(\sigma, p)$ are $(n - |\sigma|)$-acyclic then $\widetilde{X}$ is n-acyclic at infinity. If in addition, the links and punctured links of vertices are all 1-connected, then $\widetilde{X}$ is n-connected at infinity.*

In the statement $|\sigma|$ denotes the dimension of σ, the condition "(-1)-acyclic" means "non-empty" while n-acyclic for $n < -1$ is vacuous.

At first glance it appears that an infinite number of conditions have been hypothesized – since p can range over a continuum of points – thus making the theorem difficult to apply. Since every punctured link deformation retracts onto a subcomplex, however, one only needs to check the topology of finitely many finite complexes in order to check these conditions. In particular, to establish that a complex is n-acyclic at infinity, one only needs to compute the homology of finitely many finite simplicial complexes.

The property of being n-acyclic at infinity is closely related to the notion of a duality group. A group G is a *duality group* if there is a dualizing module D such that $H_i(G, M) \simeq H^{n-i}(G, M \otimes D)$ for all G-modules M. When G admits a finite $K(G, 1)$ whose dimension, n, equals the cohomological dimension of G, then G is a duality group if and only if the universal cover of this space is $(n - 2)$-acyclic at infinity. Thus, an immediate corollary to Theorem 1.1 is that a finite n-dimensional $\mathrm{CAT}(0)$ cubical complex whose links of vertices are $(n - 2)$-acyclic and remain $(n - 2)$-acyclic when one removes any simplex in the link, has a duality group as its fundamental group.

As in the cubical case, there is an immediate corollary about duality groups.

Corollary 1.4. *Let X be a finite,* $\mathrm{CAT}(0)$ *complex of dimension n. If for each cell $\sigma \subset X$, one has $Lk(\sigma)$ and $PLk(\sigma, p)$ are $(n - |\sigma| - 2)$-acyclic — for each point $p \in Lk(\sigma)$ — then $\pi_1(X)$ is a duality group.*

In Section 2 we quickly sketch the relevant definitions. The proof of Theorem 1.3 is given in Section 3 and a partial converse is established in Section 5. In between we give examples illuminating the murky connection between end connectivity and local connectivity. In particular, we highlight the difference between the punctured condition and the Cohen-Macaulay property, and show that a full converse of Theorem 1.3 cannot hold. It is, however, known to hold in some interesting cases. See [4] and [8].

Acknowledgements: We thank Mike Davis and Eric Swenson for helpful conversations during the preparation of the article.

2. Definitions

We assume the reader is somewhat familiar with CAT(0) spaces. If not, they should read [6].

Definition 2.1 (CAT(0) complexes)**.** In this paper, we take finite *locally* CAT(0) *complexes* to be finite piecewise-Euclidean or piecewise-hyperbolic polyhedral complexes satisfying the link condition (II.5.4 in [6]). A subdivision of such a complex allows us to restrict our focus to simplicial complexes where each simplex is isometric to the convex hull of a set of points in general position in $\mathbb{R}^n$ or $\mathbb{H}^n$.

Links of simplices in CAT(0) complexes are defined as they are in any simplicial complex, although in the CAT(0) setting they come with a natural piecewise spherical metric. For any point p in a CAT(0) complex its space of directions consists of a metric sphere of sufficiently small radius that the sphere is contained in the first cellular neighborhood of the simplex supporting p. The space of directions can also be given a piecewise spherical metric, viewing it as the spherical join of the unit tangent bundle at p (in the simplex supporting p) with the link of the simplex supporting p. The metric structure of the space of directions at a point p does not depend on the simplicial subdivision, although obviously the combinatorial structure does.

Definition 2.2 (Topology at infinity)**.** Topology at infinity is the study of deep topological properties of complements of compact sets, where "deep" is a technical term as in the work of Kapovich and Kleiner [11]. It suffices for this paper to let $\widetilde{X}$ be the universal cover of a finite, aspherical, simplicial complex. We say $\widetilde{X}$ is *m-connected at infinity* if given any compact $C \subset \widetilde{X}$ there is a compact $D \supset \widetilde{X}$ such that any map $\phi : S^n \to \widetilde{X} - D$ extends to a map $\widehat{\phi} : B^{n+1} \to \widetilde{X} - C$ for all $-1 \leq n \leq m$. Similarly, $\widetilde{X}$ is *m-acyclic at infinity* if any n-cycle whose support is outside of D is the boundary of an $(n+1)$-chain with support outside of C for all $-1 \leq n \leq m$. In particular, since (-1)-acyclic means "non-empty," (-1)-acyclic at infinity means "non-compact." (See [12].)

We should note that the topology at infinity of a CAT(0) group is not necessarily related to the topology of any of its visual boundaries. For example, Mike Mihalik has constructed a finite CAT(0) complex whose universal cover is 1-connected at infinity but the visual boundary is not 1-connected [13].

The property of being simply connected at infinity has many applications in the study of manifolds, and being m-acyclic at infinity is important in the study of group cohomology. In particular, the properties of being m-acyclic and m-connected at infinity are quasi-isometry invariants [5, 10].

3. The Proof

Our argument uses the function given by distance from a fixed base vertex $v \in \widetilde{X}$, which we denote by $\mathrm{D}_v : \widetilde{X} \to [0,\infty)$. Thus if $x \in \widetilde{X}$ then $\mathrm{D}_v(x) = d_{\widetilde{X}}(v,x)$. In establishing Theorem 1.3 we use the standard "Morse Lemma" as developed by Bestvina and others (see [1], [2] and [3]). In this setting a *Morse function* is a map M from a cell complex to the closed half line $[0,\infty)$ where the only critical points of M, i.e., those points where the topology of $M^{-1}(r)$ can change, occur at vertices. In the case where $\widetilde{X}$ is a $\mathrm{CAT}(0)$ cubical complex, it is always true that the critical points of the function D_v are vertices of $\widetilde{X}$ (see [4]). In the general $\mathrm{CAT}(0)$ setting this isn't the case; the topology can change whenever one first encounters a cell, which may be at a vertex or at some interior point of the cell. This is however only a minor annoyance as we explain below.

The proof of the following lemma is a standard application of the fact that Euclidean and hyperbolic simplices are convex and $\mathrm{CAT}(0)$ metrics are convex.

Lemma 3.1. *Let $\widetilde{X}$ be a contractible* $\mathrm{CAT}(0)$ *simplicial complex, v a base vertex of $\widetilde{X}$ and $\sigma \subset \widetilde{X}$ any simplex. Then there is a unique point $\sigma_v \in \sigma$ that is closer to v than any other point of σ.*

Definition 3.2 (Morse subdivision)**.** Let X be a finite, locally $\mathrm{CAT}(0)$ simplicial complex, let $\widetilde{X}$ be its universal cover, let $v \in \widetilde{X}^{(0)}$ and define D_v as above. The *Morse subdivision* of $\widetilde{X}$, denoted $\widetilde{X}_m$, is a geodesic subdivision induced by adding vertices at the critical points of D_v. We note that for this construction one doesn't need to assume that the basepoint v is a vertex. So let v be any point in $\widetilde{X}$, and begin the subdivision on the 1-skeleton of $\widetilde{X}$. Add a vertex to any critical point of D_v that occurs in the interior of an edge, and then give each such edge e a new simplicial structure by taking the cone from the point on e closest to v to the original vertices. Note that all the critical points of edges in this subdivision are vertices.

Assume that $\widetilde{X}^{(n-1)}$ has been subdivided. Let σ be an n-simplex, and let p be the point of σ that is closest to v. Cone from p to the previously subdivided $(n-1)$-skeleton of σ. (Since metric Euclidean and hyperbolic simplices are convex, this can be done geometrically.)

Given any simplex σ in $\widetilde{X}_m$, the point closest to v in $\widetilde{X}_m$ must be a vertex, since σ sits inside a simplex σ' in $\widetilde{X}$, and σ contains the point of σ' that is closest to v as a vertex.

One should note that $\widetilde{X}_m$ is not in general a $\pi_1(X)$-complex as the subdivision is not $\pi_1(X)$-equivariant. For example, if $\widetilde{X}$ is the standard $(3,3,3)$-tessellation of $\mathbb{R}^2$ by regular triangles, then $\widetilde{X}_m$ is illustrated in Figure 1. Asymptotically the Morse subdivision of $\widetilde{X}$ is the same tessellation of $\mathbb{R}^2$, but with the addition of six rays based at v.

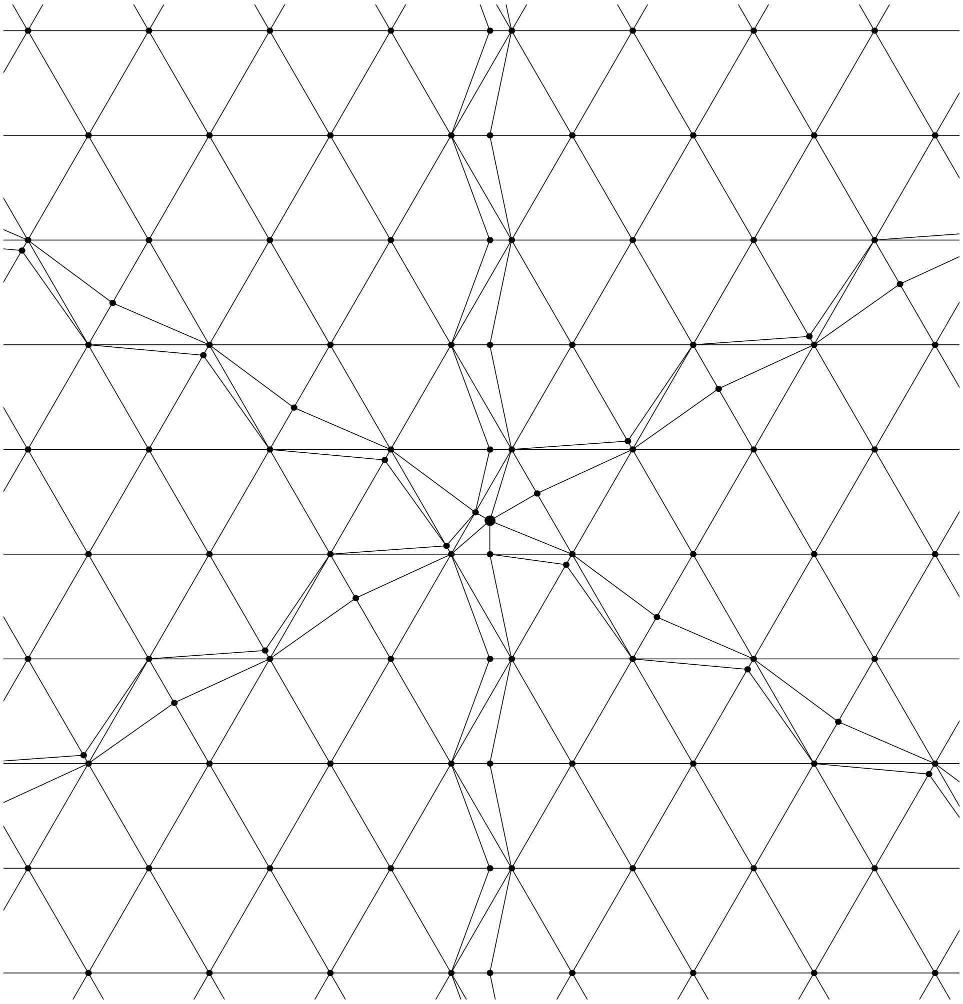

FIGURE 1. A Morse subdivision of the plane.

Lemma 3.3. *Let X be a finite, locally* $\mathrm{CAT}(0)$ *simplical complex, let $\widetilde{X}$ be its universal cover, let $v \in \widetilde{X}^{(0)}$ and define D_v as above. Then D_v is a Morse function on $\widetilde{X}_m$.*

We can now describe and use the standard Morse lemma.

Definition 3.4. Given a base vertex v and the induced Morse function $\mathrm{D}_v : \widetilde{X} \to [0, \infty)$, the *ascending link* of a vertex $w \in \widetilde{X}_m$ is the link of w in the subcomplex of $\widetilde{X}_m$ induced by vertices w' with $\mathrm{D}_v(w') > \mathrm{D}_v(w)$. In [4] the ascending link of a vertex is referred to as the "opposite part" of its link. It has also been colloquially referred to as the "dark side of the moon" where the base vertex is thought or as the "sun".

Lemma 3.5 (Morse Lemma). *Let $\widetilde{X}$ be a locally finite contractible* CAT(0) *complex, let D_v be the Morse function defined using a fixed base vertex v, and let $\widetilde{X}_m$ be the Morse subdivision of $\widetilde{X}$. If the ascending link of each vertex $w \in \widetilde{X}_m$ is n-acyclic, then $\widetilde{X}$ is n-acyclic at infinity.*

Theorem 1.3 follows immediately from Lemma 3.5 and the following lemmas.

Lemma 3.6. *Given any vertex $w \in \widetilde{X}_m$, $Lk_{\uparrow}(w)$ is a deformation retract of $PLk(w,p)$ where p is the point in $Lk(w)$ corresponding to the geodesic $[v,w]$.*

Proof. For sufficiently small $\epsilon > 0$ one can embed the link of w in $\widetilde{X}_m$ as the metric sphere of radius ϵ based at w. Let the puncture point — where the geodesic $[v,w]$ intersects the embedded link — be denoted p. Then it follows from the Side-Angle-Side definition of CAT(0) (part (5) of Proposition II.1.7 in [6]) that a point w' in the link of w is further from v than w if and only if the spherical distance from the puncture point p to w' is at least $\pi/2$. □

Lemma 3.7. *Let X be a finite,* CAT(0) *complex. If for each cell $\sigma \subset X$, and each point $p \in Lk(\sigma)$, one has $PLk(\sigma,p)$ is $(n-|\sigma|)$-acyclic, then for each vertex $w \in \widetilde{X}_m$, the ascending link of w, $Lk_{\uparrow}(w)$, is n-acyclic.*

Proof. As was mentioned in Section 2, the space of directions at any point $p \in \widetilde{X}$ is the spherical join of the unit tangent bundle at p in the simplex σ supporting p, with the spherical link of σ. Since the link of any simplex $\sigma \subset \widetilde{X}$ is presumed $(n-|\sigma|)$-acyclic, it follows that the space of directions at any point $p \in \widetilde{X}$ is n-acyclic. In particular, the full link of any vertex $w \in \widetilde{X}_m$ is n-acyclic.

If w is a vertex in $\widetilde{X}_m$ then its punctured link at any puncture point p is the same as the punctured space of directions for the point w thought of as a point in $\widetilde{X}$. Should the puncture point p be orthogonal to the cell, that is, p sits fully in the link of the simplex σ supporting $w \in \widetilde{X}$, then the punctured space of directions is the spherical join of the unit tangent bundle ($= S^{\dim(\sigma)-1}$) with the punctured link of σ (which is $(n-|\sigma|)$-acyclic by hypothesis). Should the puncture point p not be orthogonal to σ, then the punctured space of directions is homotopy equivalent to a cone over some subcomplex of the link of σ, hence it is actually contractible. □

The reader who wishes to avoid Morse theory arguments can easily establish Theorem 1.3 using local pushing arguments as in [4].

4. Examples

Example 4.1. We learned of the following example from Ross Geoghegan who attributes it to Mladen Bestvina.

Duality properties frequently occur in Cohen-Macaulay complexes. Some people have wondered if any finite $K(\pi,1)$ complex X, where the links of the vertices in X are Cohen-Macaulay, then $\pi_1(X)$ is a duality group. (Theorem

C of [4] can be cited as giving evidence for such a guess.) However, the following example shows that this reasonable guess is false. One takes X to be two tori, cellated as indicated in Figure 2, joined along a common edge e.

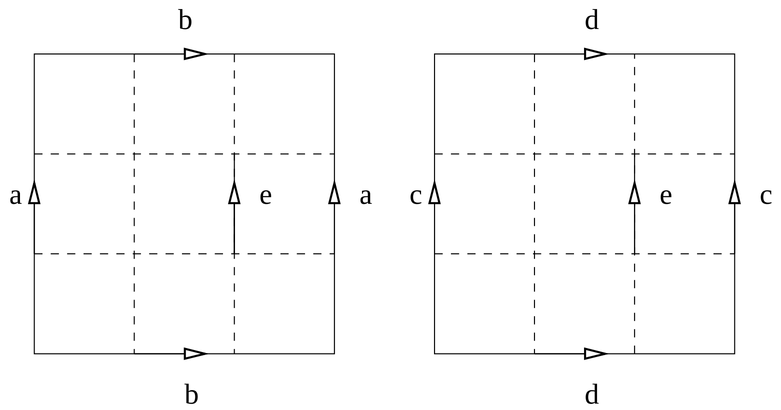

FIGURE 2. Two cubical tori attached along the edge e

The links of all but the two vertices bounding the shared edge e are circles; the links of the two vertices bounding the common edge are two-petal roses. So all links in X are Cohen-Macaulay. But the fundamental group of X is $\mathbb{Z}^2 * \mathbb{Z}^2$ which is a two-dimensional group that is not one-ended, hence it's not a duality group.

The joined tori complex X admits a CAT(0) cubical metric, and it's instructive to reconsider this example in light of Theorem 1.1. Given any finite, 2-dimensional CAT(0) cubical complex X, in order to show that $\pi_1(X)$ is a duality group via this theorem, one would need to know that the link of each vertex is 0-acyclic and that these links remain 0-acyclic when one removes any simplex from the link. In other words, the links are connected graphs and these graphs contain no cut vertices or edges. In the case of the joined tori, the link of a vertex bounding the shared edge of X is a two-petal rose, which has a cut vertex. This example highlights that the introduction of punctured links in Theorem 1.3 is necessary.

Example 4.2. On the other hand, one can't hope for a full converse to Theorem 1.3. Examples in [8] show that a space can be highly connected at infinity without having highly connected punctured links. Those examples are in the context of Coxeter group actions. Here we outline another example that fits well with the presentation in this paper.

Consider the group $G = \langle x, y, z \mid txyt^{-1}y^{-1}x \rangle$, the fundamental group of the non-orientable surface with Euler characteristic -1. The universal cover of the presentation 2-complex is the hyperbolic plane $\mathbb{H}$ — tessellated by hexagons meeting six at a vertex — so G is obviously one-ended.

The presentation 2-complex of G is gotten by taking a hexagon and making the edge identifications as indicated by the defining relation for G. One can subdivide this hexagon into 6 quadrilaterals by coning the midpoints of edges to baricenter of hexagon. Let $\mathbb{Z}_2$ act on the hexagon by rotating an interior, concentric hexagon through an angle of π, and let K be the quotient complex. Equivalently, start with a Möbius band decomposed into six rectangles so that the boundary is a cycle of length 6 and the equator is a cycle of length 3. Attach a triangle to the equator to form a cell complex homotopy equivalent to a disk (the h.e. is defined by shrinking the triangle to a point). Label the six edges edges on the boundary of the original Möbius band as in the presentation for G, and make the corresponding edge identifications. By shrinking the interior triangle one sees that this complex K is homotopy equivalent to the presentation 2-complex for G, hence $\pi_1(K) = G$ and $\widetilde{K}$ is homotopy equivalent to $\mathbb{H}$.

The complex K admits a $\mathrm{CAT}(0)$ structure. The interior triangle is given the metric of an equilateral triangle, and each rectangle is given the metric of a quadrilateral with angles $5\pi/6$ along the inner triangle and $\pi/6$ along the boundary of the Möbius band. To establish the local non-positive curvature one simply has to make sure that no links of vertices have short loops. Due to the edge identifications, the six vertices along the boundary of the Möbius band are all identified, and the link of this vertex is a cycle of combinatorial length 12, each edge having length $\pi/6$. The links of the three vertices on the interior triangle are all Θ-graphs, with the edge coming from the triangle having length $\pi/3$ and the top and bottom arcs having lengths $5\pi/3$ each.

While $\widetilde{K}$ is one-ended, and even in fact has a single circle at infinity, the punctured links of vertices on the interior triangle are not connected; just puncture along the short arc in the θ-graph.

We note that there is nothing special about our choice of surface group in this example; the same techniques apply to any (hyperbolic) surface group.

5. Semistability

As the examples in Section 4 have shown, a full converse to Theorem 1.3 is not possible. There is, however, an important partial converse which is established in Corollary 5.3 below.

Definition 5.1. Let $\widetilde{X}$ be a contractible, locally finite complex, and let $C_0 \subset C_1 \subset C_2 \subset \cdots$ be a nested, properly increasing sequence of compact subcomplexes of $\widetilde{X}$ that exhaust $\widetilde{X}$. One may then look at the induced sequences of homology groups

$$H_n(\widetilde{X} - C_0) \leftarrow H_n(\widetilde{X} - C_1) \leftarrow H_n(\widetilde{X} - C_2) \leftarrow \cdots .$$

This system is H_n-*semistable* if for each n there is a value $f(n) > n$ such that the image of $H_n(\widetilde{X} - C_k) \to H_n(\widetilde{X} - C_n)$ is independent of k for any $k \geq f(n)$.

The homotopy version of semistability is defined in a similar fashion, looking at the induced sequence of homotopy groups

$$\pi_n(\widetilde{X} - C_0) \leftarrow \pi_n(\widetilde{X} - C_1) \leftarrow \pi_n(\widetilde{X} - C_2) \leftarrow \cdots .$$

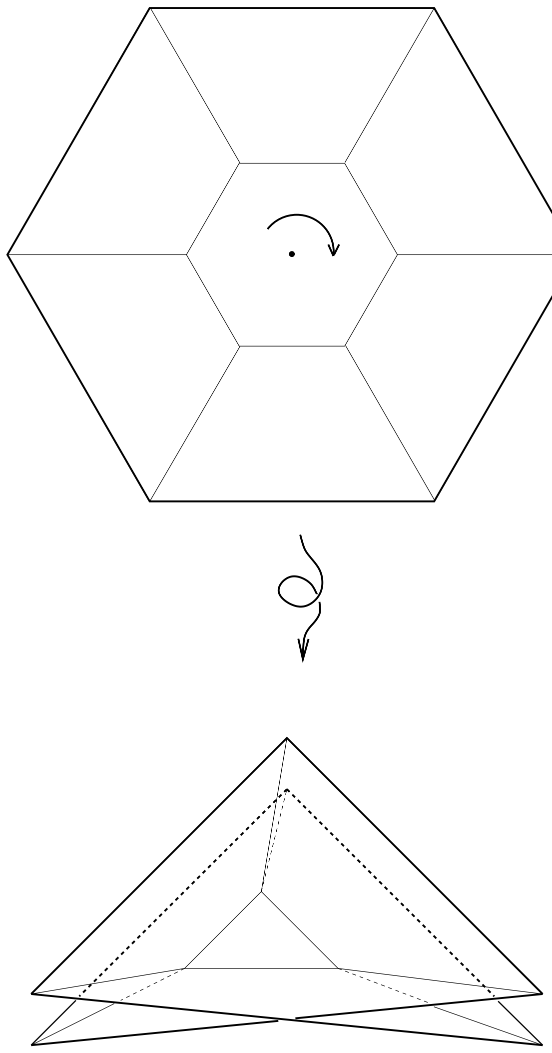

FIGURE 3. The complex K

When considering homotopy groups one needs to be careful with basepoints and this can, in general, be a significant problem. In order to chose basepoints one uses a proper ray $\rho : [0, \infty) \to \widetilde{X}$ such that $\rho([i, \infty)) \subset \widetilde{X} - C_i$ for all i, and then uses $\rho(i)$ as the basepoint for $\pi_1(\widetilde{X} - C_i)$. In general the resulting inverse limit of fundamental groups can depend on ones choice of basepoint ray; it is unknown if this pathology can exist in the universal cover of a finite $K(\pi, 1)$. It is however known that if the sequence of fundamental groups is semistable for any basepoint ray, then it is semistable for all of them, and the inverse limit is independent of choice of base ray.

We let $(\widetilde{X}_m)_{>r}$ be the maximal subcomplex of $\widetilde{X}_m - B_r(v)$ where v is the base vertex and $B_r(v) = \{x \in \widetilde{X}_m \mid d(v, x) \leq r\}$. Equivalently, $(\widetilde{X}_m)_{>r}$

is the maximal subcomplex contained in $\mathrm{D}_v^{-1}((r,\infty))$. The Morse theory argument of the previous section establishes that $(\widetilde{X}_m)_{>r}$ is n-acyclic when the links and punctured links of vertices in $\widetilde{X}_m$ are n-acyclic. The same hypotheses also imply semistability.

Theorem 5.2. *Let X be a finite, locally $\mathrm{CAT}(0)$ complex. If for each cell $\sigma \subset X$, and each point $p \in Lk(\sigma)$, one has $Lk(\sigma)$ and $PLk(\sigma, p)$ are $(n-|\sigma|)$-acyclic, then $\widetilde{X}$ is H_{n+1}-semistable at infinity.*

Similarly, if for each cell $\sigma \subset X$, and each point $p \in Lk(\sigma)$, one has $Lk(\sigma)$ and $PLk(\sigma, p)$ are $(n-|\sigma|)$-connected, then $\widetilde{X}$ is π_{n+1}-semistable at infinity.

Notice in particular that when the links and punctured links are connected, the complex is π_1-semistable.

Proof. First consider the case when $n = 0$, in which case our hypotheses imply that the ascending links of vertices in $\widetilde{X}_m$ are connected, and hence by Theorem 1.3, all the $(\widetilde{X}_m)_{>r}$ are connected. Let $\phi : S^1 \to (\widetilde{X}_m)_{>r}$ be a continuous map, which we may assume is cellular. After sufficient subdivision we may further assume that S^1 is a simplicial circle such that ϕ is actually a combinatorial map, taking vertices to vertices and edges to edges. Let w be a point in S^1 such that $\mathrm{D}_v(\phi(w))$ is minimal. Since we are in the Morse subdivision $\widetilde{X}_m$, and ϕ is a combinatorial map, w is a vertex. Let u_1 and u_2 be the vertices in S^1 that are adjacent to w.

The vertices $\phi(u_i)$ correspond to vertices in $\mathrm{Lk}_\uparrow(w)$, and hence by hypothesis there is an edge path in $\mathrm{Lk}_\uparrow(w)$ joining u_1 to u_2. We may then construct a new function $\phi' : S^1 \to (\widetilde{X}_m)_{>r}$ such that $\phi' = \phi$ except on the interval $[u_1, u_2]$ where ϕ' maps to a path in $\mathrm{Lk}_\uparrow(w)$. Repeated application of this local pushing process shows that $\phi(S^1)$ is homotopic to loops in $(\widetilde{X}_m)_{>R}$ for any $R > r$, and hence the inverse system

$$\pi_1((\widetilde{X}_m)_{>0}) \leftarrow \pi_1((\widetilde{X}_m)_{>1}) \leftarrow \pi_1((\widetilde{X}_m)_{>2}) \leftarrow \cdots$$

induced by the filtration $v = B_0(v) \subset B_1(v) \subset B_2(v) \subset \cdots$ is semistable.

We now consider the homological claim, assuming that links and punctured links of simplices are $(n-|\sigma|)$-acyclic. Let z be an $(n+1)$-cycle with $\mathrm{supp}(z) \subset (\widetilde{X}_m)_{>r}$. Let w be a point in $\mathrm{supp}(z)$ that minimizes the distance to v; w is necessarily a vertex in $\widetilde{X}_m$. Since w is of minimum distance, $\mathrm{supp}(z) \cap \mathrm{Lk}_\uparrow(w)$ is an n-cycle in $\mathrm{Lk}_\uparrow(w)$, and since $\mathrm{Lk}_\uparrow(w)$ is n-acyclic, there is an $(n+1)$-chain z_w in $\mathrm{Lk}_\uparrow(w)$ whose boundary is $\mathrm{supp}(z) \cap \mathrm{Lk}_\uparrow(w)$. Thus if we set

$$z' = z - z|_{\mathrm{St}(w)} + z_w$$

then z and z' are homologous $(n+1)$-cycles. Repeated application of such local replacements show that z is homologous to $(n+1)$-cycles in $(\widetilde{X}_m)_{>R}$ for any $R > r$.

To establish the homotopy result, note that by the Hurewitz theorem it suffices to establish that $(\widetilde{X}_m)_{>r}$ is 1-connected and n-acyclic. ∎

This semistability result allows us to establish a partial converse to Theorem 1.3. For example, let X be a finite CAT(0) complex where the punctured links are all connected, but there is a vertex v whose full link is not simply connected. Let v also denote a chosen lift of v to $\widetilde{X}_m$. Then the filtration by closed metric balls $v = B_0(v) \subset B_1(v) \subset B_2(v) \subset \cdots$ induces an inverse system

$$\pi_1((\widetilde{X}_m)_{>0}) \leftarrow \pi_1((\widetilde{X}_m)_{>1}) \leftarrow \pi_1((\widetilde{X}_m)_{>2}) \leftarrow \cdots$$

where the bonds are surjections. The deformation retraction along geodesics shows that $\pi_1((\widetilde{X}_m)_{>0}) \simeq \pi_1(\mathrm{Lk}(v)) \neq 1$ and hence the inverse limit of this system is non-trivial. Thus $\widetilde{X}_m$ is not simply connected at infinity.

Corollary 5.3. *Let X satisfy the conditions of Theorem 1.3, so that we know X is n-acyclic at infinity. If there is a simplex $\sigma \subset X$ such that $H^{n+1-|\sigma|}(Lk(\sigma)) \neq 0$, then $\widetilde{X}$ is not $(n+1)$-acyclic at infinity.*

Similarly, if the punctured links are all connected, but the link of any vertex is not simply connected, then $\widetilde{X}$ is not simply connected at infinity.

For example, if X is a compact, non-positively curved 3-dimensional pseudomanifold, which is not a manifold, then the fundamental of some vertex link is non-trivial, so by Corollary 5.3, $\pi_1^\infty(\widetilde{X})$ is non-trivial.

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