Two-dimensional Artin groups with CAT(0) dimension three *

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Abstract

We exhibit 3-generator Artin groups which have finite 2-dimensional Eilenberg-Mac Lane spaces, but which do not act properly discontinuously by semi-simple isometries on a 2-dimensional CAT(0) complex. We prove that infinitely many of these groups are the fundamental groups of compact, non-positively curved 3-complexes. These examples show that the geometric dimension of a CAT(0) group may be strictly less than its CAT(0) dimension.

One way of associating a dimension to a discrete group G is to take the minimal dimension of a contractible free G-CW complex or, equivalently, to take the minimal dimension of a $K(G,1)$ complex. This is called the geometric dimension of G. For example, groups with torsion are infinite dimensional, and the dimension of $\mathbb{Z}^n$ is equal to n. Contractible free G-CW complexes arise naturally in geometric group theory, where the contractibility of the G-CW complex is often a consequence of non-positive curvature. Specific instances include the case when G acts freely and properly discontinuously by isometries on a CAT(0) complex, and the case when a torsion free word-hyperbolic G acts on its Rips complex.

We define the *CAT(0) dimension* of G to be the minimal dimension of a CAT(0) CW-complex on which G acts properly discontinuously by semi-simple isometries. (See section 1 for precise definitions of "CAT(0)" and "semi-simple"). The CAT(0) dimension is taken to be ∞ if there exists no such action on a finite dimensional complex. Clearly the geometric dimension of a torsion free group is bounded above by its CAT(0) dimension. In the case of groups with torsion it is

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more appropriate to compare the CAT(0) dimension of G with the minimal dimension of a proper G-space. This is done in [2] where dimension gaps are found for certain quotients of hyperbolic triangle groups.

Note that the classical definition of geometric dimension does not require that the quotient complex be compact: for example, an infinite rank free group has dimension one. Similarly, the condition that the action on X be semi-simple allows for non-compact quotients, requiring only that every free homotopy class in the quotient X/G has a minimal length closed geodesic representative, and is strictly weaker than 'cocompact'. This condition also has the advantage that semi-simple actions are still semi-simple when restricted to subgroups; something which is not necessarily true for cocompact actions.

In this paper we exhibit an infinite family of Artin groups which all have geometric dimension 2, but have CAT(0) dimension 3. These examples illustrate an important point for researchers who endeavour to prove that certain families of groups are CAT(0); namely, one should not just try to put a non-positively curved metric on a minimal dimensional $K(G,1)$, but should be prepared to work with higher dimensional $K(G,1)$'s for a given group G.

Our examples come from the following family. Let m, n and p be integers which are all at least 2. We denote by $A(m,n,p)$ the three generator Artin group with the following presentation

$$\langle\, a,b,c \mid (a,b)_m = (b,a)_m,\ (b,c)_n = (c,b)_n,\ (a,c)_p = (c,a)_p \,\rangle$$

where $(a,b)_m$ denotes the alternating product of length m of a's and b's, beginning with a.

There are two key facts which we shall establish about these Artin groups. The first is that 3 is a lower bound for the CAT(0) dimension in the case where $p = 2$ and m and n are both odd.

Theorem A. *Let m, n be odd integers ≥ 3. The Artin group $A(m,n,2)$ does not act properly discontinuously by semi-simple isometries on any 2-dimensional CAT(0) complex.*

Note that T. Brady and J. McCammond [4] have constructed locally CAT(0) 2-dimensional Eilenberg-Mac Lane complexes for all $A(m,n,p)$ where all $m,n,p \geq 3$, as well as those 3-generator Artin groups with fewer than 3 Artin relations. It is not known whether any of the groups $A(m,2n,2)$ have CAT(0) dimension 2.

The second key fact is that there are explicit 3-dimensional CAT(0) structures for most of the $A(m,n,2)$ groups.

Theorem B. *All but finitely many of the 3-generator Artin groups $A(m,n,2)$ are the fundamental groups of compact nonpositively curved (i.e: locally CAT(0)) 3-dimensional piecewise Euclidean complexes.*

The methods used to prove Theorem B can also be used to produce 3-dimensional CAT(0) structures for other classes of Artin groups – see Theorem 4.4.

From [8] we know that the Artin groups $A(m,n,p)$ have geometric dimension 2 precisely when $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} \leq 1$. Combining this fact with Theorems A and B we get that infinitely many of the $A(m,n,2)$ have geometric dimension 2, but CAT(0) dimension 3. Other examples which distinguish between the geometric and CAT(0) dimensions are given by M. Bridson in [6].

In our definition of CAT(0) dimension we talk of an "n-dimensional CAT(0) CW-complex" by which we mean a CAT(0) space in which the underlying topology is that of an n-dimensional CW-complex. However, in the proof of Theorem A we do not make any use of the CW-structure. Thus we have the slightly stronger statement (Theorem 3.1) in which the word "complex" is replaced by "space". The appropriate notion of "dimension", for an arbitrary CAT(0) space, is that of covering dimension (see [10] for the general theory).

This paper is organized as follows. In Section 1 we recall basic facts about minsets of isometries of CAT(0) spaces. The dihedral type Artin groups are introduced in Section 2. Proposition 2.2 of Section 2 is used in the proof of Theorem A to which Section 3 is devoted. In Section 4 we introduce compact, non-positively curved, 3-complexes for the dihedral type Artin groups and use these as building blocks to construct compact, non-positively curved 3-dimensional $K(\pi, 1)$ complexes for a large family of Artin groups, including those of Theorem B. Section 5 closes with some remarks and questions that arose at various stages during the preparation of this paper.

1 Minsets and periodic flats in CAT(0) spaces

In this section we recall some standard facts about discrete groups of semi-simple isometries of CAT(0) spaces. We shall state the flat torus theorem, as well as some facts about the structure of minsets of infinite order semi-simple isometries. These facts will be used in the proof of the Proposition 2.2, which is a key ingredient in the proof of Theorem A. We begin by introducing the notion of a CAT(0) space.

A *geodesic* between points x and y in a metric space X is an isometric map of the closed interval $[0, d(x,y)]$ into X such that the endpoints are mapped to x and y. We often denote the image of such a geodesic by $[x, y]$. A *geodesic triangle* $\triangle(x,y,z)$ in a metric space X is simply a union of three geodesics $[x,y] \cup [y,z] \cup [z,x]$. A *comparison triangle* for $\triangle(x,y,z)$ is a triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ in the Euclidean plane $M^2_1 = \mathbb{E}^2$ so that $d_X(x,y) = d_{\mathbb{E}^2}(\overline{x}, \overline{y})$, $d_X(y,z) = d_{\mathbb{E}^2}(\overline{y}, \overline{z})$, and $d_X(z,x) = d_{\mathbb{E}^2}(\overline{z}, \overline{x})$. Suppose $p \in [x,y] \subset \triangle(x,y,z)$. A *comparison point* for p is a point $\overline{p} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ such that $\overline{p} \in [\overline{x}, \overline{y}]$ and $d_X(p,x) = d_{\mathbb{E}^2}(\overline{p}, \overline{x})$. A geodesic triangle $\triangle(x,y,z)$ in a metric space X is said to *satisfy the CAT(0) inequality* if for all $p, q \in \triangle(x,y,z)$ and comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ we have $d_X(p,q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q})$. A geodesic metric space X (one in which there is a geodesic path between any pair of points) is said to be a *CAT(0) space* if all geodesic triangles in X satisfy the CAT(0) inequality. (One defines the notion of a *CAT(1) space* similarly by comparing every geodesic triangle of total perimeter less than 2π with its comparison triangle in the standard 2-sphere $M^2_1 = \mathbb{S}^2$.)

The triangle inequality may be usefully rephrased in terms of the Alexandrov (or upper) angle between nontrivial geodesic segments issuing from a common point (see [7], p.9, for a definition): (X, d) is a CAT(κ) space ($\kappa = 0$ or 1) if, for every geodesic triangle Δ with distinct vertices (and of perimeter less than 2π in the case $\kappa = 1$), the Alexandrov angle between any two sides is no greater than the corresponding angle in the comparison triangle Δ' in M^2_κ.

A space is *locally* CAT(κ) if every point lies in an open metric ball on which the induced metric is CAT(κ). The Cartan-Hadamard Theorem (see [7]) states that (X, d) is CAT(0) if and only if it is locally CAT(0) and simply connected.

An important source of examples of CAT(κ) come from metric simplicial complexes. These are complexes which are obtained by glueing together metric simplices of constant Riemannian curvature (in our case, either Euclidean or spherical simplices) by isometries of their faces. If one builds X from a finite number of isometry types of cells ("finitely many shapes") then a result of Bridson [7] ensures that the intrinsic pseudometric on X (which extends the metric on each simplex of X) is in fact a complete geodesic metric.

We will make use of the following criterion for determining whether or not a given piecewise Euclidean complex (with finitely many shapes) is locally CAT(0) (see [9] Ch 10 Theorem 15 or [7], Theorem II.5.4). A piecewise Euclidean or spherical complex X is said to satisfy the *link condition* if for every vertex $v \in X$ the piecewise spherical link complex $\mathrm{Lk}(v, X)$ is CAT(1). A piecewise Euclidean complex with finitely many shapes is locally CAT(0) if and only if it satisfies the link condition. A piecewise spherical complex (for instance a link complex) is CAT(1) if and only if it satisfies the link condition and contains no closed geodesics (isometrically embedded circles) of length less than 2π. Note that a 1-dimensional complex (a metric graph) is CAT(1) if and only if every embedded circle has length at least 2π.

In the case of a spherical link complex $\mathrm{Lk}(v, X)$, we distinguish the intrinsic metric d from the metric $d^{(\pi)}(x, y) = \min\{d(x, y), \pi\}$ which measures the Alexandrov angle between geodesic segments emanating from v.

For any point v in a metric space X, the *space of directions* $S_v(X)$ is defined as the the set of equivalence classes of geodesic segments emanating from x where two segments are equivalent if the Alexandrov angle between them is 0 (see [7], II.3.18). This is a metric space, with the metric induced by the Alexandrov angle, and corresponds to the link $\mathrm{Lk}(v, X)$ with the metric $d^{(\pi)}$ in the case that X is a piecewise Euclidean complex (or any polyhedron with cells of constant curvature).

We shall work with minsets of isometries of CAT(0) spaces in Sections 2 and 3. Definition 1.1 through Theorem 1.3 establish all the notation and background facts that we shall use.

Definition 1.1. Let g be an isometry of a metric space X. The *translation length of* g, denoted by $\ell(g)$, is defined to be

$$\ell(g) \;=\; \inf\{d(x, gx) \mid x \in X\}\,,$$

and the *minset of* g, denoted by $Min(g)$, is defined to be the possibly empty set

$$Min(g) \;=\; \{x \in X \mid d(x, gx) = \ell(g)\}\,.$$

An isometry of a CAT(0) space is called *semi-simple* if it has a nonempty minset. If g is semi-simple with nonzero translation length then we say that g is *hyperbolic*. In this case any g-invariant geodesic line in X shall be called an *axis of* g or g*-axis*. By [7] Proposition II.6.2, the g-axes all lie in $Min(g)$.

It is a theorem that a finite order isometry of a CAT(0) space X has a fixed point (see [7] Corollary II.2.8, for instance). So, if a group acts properly discontinuously by semi-simple isometries on X then it is clear that group elements have infinite order if and only if they are hyperbolic.

We now summarize some fundamental properties of individual isometries and their minsets which may be found in [7], Chapter II.6 (see also [5], Proposition 2.13).

Let g be a semi-simple isometry of a CAT(0) space X. Then $Min(g)$ is always a closed, convex subspace of X. If $\ell(g) \neq 0$ then $Min(g)$ is isometric to the metric product of $\mathbb{R}$ with a CAT(0) space Y, where each $\mathbb{R}$-fibre is an axis of g. The element g acts on $Min(g)$ by translating along the $\mathbb{R}$ factor and fixing the Y factor pointwise. Moreover, each isometry of X which commutes with g leaves $Min(g)$ invariant and its restriction to $Min(g)$ also respects the product structure.

In the 2-dimensional CAT(0) setting the structure of minsets is particularly simple. We record this structure in the next Propostion, which follows immediately from the preceding paragraph and Lemma 3.2 of [6].

Proposition 1.2. *Let X be a CAT(0) space of covering dimension 2, and let g be a hyperbolic isometry of X. Then*

1. *$Min(g)$ is the product of $\mathbb{R}$ and an $\mathbb{R}$-tree Y, where each fibre $\mathbb{R} \times \{y\}$ is an axis of g.*
2. *The space of directions $S_v(Min(g))$ of any point $v \in Min(g)$ is the orthogonal join of a 0-sphere $S^0 = \{g^+, g^-\}$ (namely, its link in the axis of g through v) with a (possibly empty) discrete set. In particular, a tripod graph can only be a subspace of $S_v(Min(g))$ if its valence three vertex is either g^+ or g^-.*

Our proof of Theorem A proceeds by examining a particular configuration of periodic flat planes in a candidate CAT(0) space for the Artin group. Theorem 1.3 asserts the existence of such periodic flats.

Theorem 1.3 (Flat torus theorem: [7] and [5], Theorem 2.18). *Let A be a free abelian group of rank m acting properly discontinuously by semi-simple isometries on a CAT(0) space X. Then $Min(A) = \cap_{a \in A} Min(a)$ is a convex A-invariant subspace of X which splits as a metric product $Y \times \mathbb{E}^n$, where each $a \in A$ acts by the identity on the Y factor and by a translation on the $\mathbb{E}^n$ factor. Moreover, the quotient of each $\{y\} \times \mathbb{E}^n$ by A is an n-torus.*

For example, if $\langle a, b \rangle$ is a rank 2 free abelian group then $Min(a) \cap Min(b)$ is a union of isometrically embedded flat planes on each of which $\langle a, b \rangle$ acts cocompactly as a group of translations. In the case that X is a CAT(0) space of covering dimension 2 the space Y consists of a single point. (If Y has two distinct points then it contains a geodesic segment I, and hence X contains a closed subset $\mathbb{E}^2 \times I$ of covering dimension 3, a contradiction.) Thus there is only one such flat plane. In this case we shall write

$$\Pi(a, b) = Min(a) \cap Min(b)$$

for this plane. We make the following important remark that, if there exists ANY isometrically embedded copy of $\mathbb{E}^2$ (resp. $\mathbb{E}^n$) which is invariant under $\langle a, b \rangle$ (resp. A) then, by [7] Proposition II.6.2, this plane must lie in $Min(a) \cap Min(b)$ (resp. $Min(A)$). Thus, when X is a 2-dimensional CAT(0) space, $\Pi(a, b)$ really is the unique $\langle a, b \rangle$-invariant flat plane in X.

The next lemma lists the possible configurations of periodic flats in the minset of a semi-simple isometry in a 2-dimensional CAT(0) space.

Lemma 1.4. *Let G be a group which acts properly discontinuously by semi-simple isometries on a CAT(0) space X of covering dimension 2. Suppose that $\Pi_1 \neq \Pi_2$ are two periodic planes contained in $Min(g)$, which correspond to free abelian subgroups $\langle h_1, g \rangle$ and $\langle h_2, g \rangle$ of G respectively. Then the convex closure of $\Pi_1 \cup \Pi_2$ is one of the following types.*

1. *(H-type or strip-type) In this case $\Pi_1 \cap \Pi_2 = \emptyset$ and the convex closure is a quotient space of the disjoint union of Π_1, Π_2 and a strip (product of a closed interval and $\mathbb{R}$) where one endpoint of the interval times $\mathbb{R}$ is identified with a g-axis in Π_1, and the other endpoint of the interval times $\mathbb{R}$ is identified with a g-axis in Π_2.*

2. *(X-type) In this case $\Pi_1 \cap \Pi_2$ is a g-axis and the convex closure is just the union $\Pi_1 \cup \Pi_2$.*

3. *(Band-type) In this case $\Pi_1 \cap \Pi_2$ consists of a closed interval times $\mathbb{R}$, foliated by g-axes, and the convex closure is again just the union $\Pi_1 \cup \Pi_2$.*

Proof. Since g is a hyperbolic isometry of X (it has infinite order in G and acts semi-simply), part (1) of Proposition 1.2 ensures that $Min(g)$ is the product of $\mathbb{R}$ and an $\mathbb{R}$-tree Y. Each of the planes Π_i is contained in $Min(g)$ and so inherits a product structure as $\mathbb{R}$ times an axis, A_i, of the $\mathbb{R}$-tree Y. Since $\Pi_1 \neq \Pi_2$ we have that $A_1 \neq A_2$.

There are only three possible cases for such axes. In the first case, the axes don't intersect. The convex closure of these axes consists of the union of the two axes together with a unique segment in Y realizing the distance between these axes. In the second case, the two axes intersect in a point in Y, and so their convex closure is equal to their union. In the third case, the axes intersect along a common interval, and again their convex closure is equal to their union. Now take the product of each of these pictures with $\mathbb{R}$ to get the three situations described in the statement above.

A priori, there is a fourth possible configuration for A_1 and A_2. They may share a common end. In this case, we can see that $\langle h_1, h_2 \rangle$ does not act properly discontinuously on Y, and hence that $\langle h_1, h_2, g \rangle$ does not act properly discontinuously on X. This contradicts the proper discontinuity of G. $\square$

2 The dihedral type Artin groups

In this section we describe the dihedral type Artin groups. We consider a certain pair of rank 2 abelian subgroups of a dihedral type Artin group, and give a proposition which describes how the unique flat planes invariant under these abelian subgroups are arranged in any 2-dimensional CAT(0) space on which the whole group acts properly discontinuously by semi-simple isometries. This information will be used in our proof of Theorem B.

We begin with some notation which is used to describe the basic Artin relation. Given an integer $m \geq 2$ and two letters a and b, let $(a, b)_m$ denote the alternating product $aba\ldots$ of length m in a and b. The basic Artin relation is of the form

$$(a, b)_m \;=\; (b, a)_m \,.$$

Given an integer $m \geq 2$, define the dihedral type Artin group $A(m)$ to be

$$A(m) \;=\; \langle\, a, b \mid (a, b)_m = (b, a)_m \,\rangle \,.$$

One easily sees that $A(2) = \mathbb{Z}^2$. For $m \geq 3$, the group $A(m)$ is best understood by factoring out the center Z as follows. Let $\Delta = (a, b)_m$. There are two cases to consider depending on the parity of m.

- When m is odd: $Z \;=\; \langle \Delta^2 \rangle$ and $A(m)/Z \;=\; \langle ab \rangle * \langle \Delta \rangle \;=\; \mathbb{Z}_m * \mathbb{Z}_2$. In this case we have $\Delta a = b\Delta$ and $\Delta b = a\Delta$.
- When m is even: $Z \;=\; \langle \Delta \rangle$ and $A(m)/Z \;=\; \langle a \rangle * \langle ab \rangle \;=\; \mathbb{Z} * \mathbb{Z}_{m/2}$.

Lemma 2.1. *Suppose that a and b are hyperbolic isometries of an $\mathbb{R}$-tree T, such that their product ab has a fixed point. Then the translation axes γ_a for a and γ_b for b contain in their intersection a closed interval J whose length is equal to $min\{\ell(a), \ell(b)\}$. Moreover, a and b translate in opposite directions, as determined on the interval J. (If $\ell(a) \neq \ell(b)$ then in fact $\gamma_a \cap \gamma_b = J$.)*

Proof. We use the fact that, if g is a hyperbolic automorphism of T and x any point in T, then $[x, g(x)]$ intersects γ_g in a subinterval $[x_g, g(x_g)]$ where x_g denotes the projection of x to γ_g. In particular note that $[x_g, g(x_g)]$ is symmetrically placed: $d(x, x_g) = d(g(x), g(x_g))$.

Since $\ell(ab) = 0$, we may suppose that $x \in T$ is fixed by ab. Let y denote the point $a^{-1}(x) = b(x)$. Then $[x, y]$ contains intervals $[x_a, y_a] = [x, y] \cap \gamma_a$ of length $\ell(a)$, and $[x_b, y_b] = [x, y] \cap \gamma_b$ of length $\ell(b)$. Since they are symmetrically placed, one of these two intervals, call it J, is contained in the other, and hence in $\gamma_a \cap \gamma_b$. Clearly J has length $\min\{\ell(a), \ell(b)\}$. Also, it is clear that the direction of translation of a on γ_a and that of b on γ_b induce opposing directions on the interval J. □

Consider the group $A(m)$. Let $z_{a,b}$ denote the generator (Δ or Δ^2) of the centre Z of $A(m)$. The following Proposition stems from the existence of rank 2 abelian subgroups $\langle a, z_{a,b}\rangle$ and $\langle b, z_{a,b}\rangle$ in $A(m)$.

Proposition 2.2. *Let $m \geq 3$ and suppose that the dihedral type Artin group $A(m)$ (with Artin generators a and b) acts properly discontinuously by semi-simple isometries on a CAT(0) space X of covering dimension 2. Let $z_{a,b}$ denote the central element of $A(m)$, and define the set*

$$B(a,b) := \Pi(a, z_{a,b}) \cap \Pi(b, z_{a,b}) = Min(a) \cap Min(z_{a,b}) \cap Min(b) .$$

Then

1. *$B(a,b) \cong I \times \mathbb{R}$ where $I = [p,q]$ is a nontrivial closed interval and each $\mathbb{R}$-fibre is a $z_{a,b}$-axis. The a-axes intersect the band $B(a,b)$, transversely to the $z_{a,b}$-axis, in a family of parallel line segments which we call* a-segments*. Similarly, the b-axes intersect $B(a,b)$ in a family of* b-segments*.*

Moreover, in the case that m is odd, we have the following (See Figure 1):

2. *The elements $ab, ba \in A(m)$ leave opposite edges of the band $B(a,b)$ invariant. More precisely, $\ell = \{p\} \times \mathbb{R}$ is a ba-axis and $\ell' = \{q\} \times \mathbb{R}$ is an ab-axis. Moreover, $a(\ell)$ is an ab-axis which lies in $B(a,b)$ and $b(\ell')$ a ba-axis also lying in $B(a,b)$.*

3. *For $g, h \in \{a, b, z_{a,b}\}$ write $\theta(g,h)$ for the angle in $B(a,b)$ between a positively oriented g-axis and a positively oriented h-axis. Then we have*

$$0 < \theta(a,b) = \theta(a, z_{a,b}) + \theta(b, z_{a,b}) < \pi \quad \textit{and} \ \theta(a, z_{a,b}) = \theta(b, z_{a,b}) .$$

   In particular, the a-segments are transverse to the b-segments.

Proof. By Proposition 1.2 we have $Min(z_{a,b}) \cong \mathbb{R}\times T$ where T is an $\mathbb{R}$-tree. Now $A(m)/Z$ acts by isometries on T. This action is properly discontinuous and semi-simple (by Propositions II.6.10(4) and II.6.9 of [7]). Since a and b are infinite order elements of $A(m)/Z$ they must be hyperbolic, while ab is of finite order in $A(m)/Z$ and hence fixes a point in T. Thus we may apply Lemma 2.1. Using the uniqueness of periodic 2-flats in the 2-dimensional space X, we may identify the plane $\Pi(a, z_{a,b})$ with $\gamma_a \times \mathbb{R}$ and $\Pi(b, z_{a,b})$ with $\gamma_b \times \mathbb{R}$. Note that $\gamma_a \neq \gamma_b$ since $A(m)/Z$ is never virtually $\mathbb{Z}$. Thus (1) follows by Lemma 2.1 and Lemma 1.4 (Case 3). In the case that m is odd, the element Δ conjugates a to b. It follows that a and b have equal translation lengths on both X and T. They translate in opposite directions in T and it follows from Lemma 2.1 that one endpoint of I is fixed by ab, and the other by ba. Part (2) of the Proposition now follows easily, as does part (3) with the observation that that ba translates a positive distance along ℓ in the same direction as $z_{a,b}$. $\square$

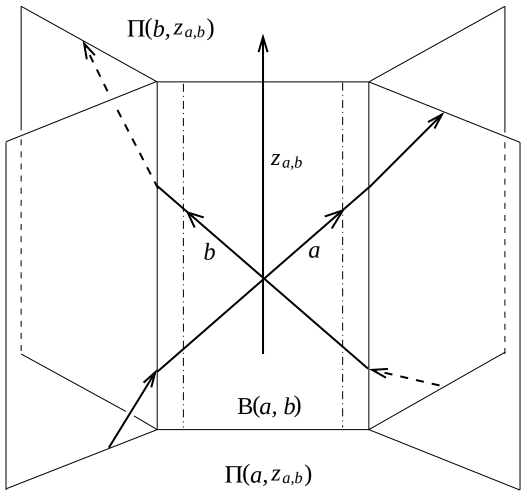

Figure 1: Planes $\Pi(a, z_{a,b})$ and $\Pi(b, z_{a,b})$ intersect in a band $B(a, b)$.

3 A lower bound for the CAT(0) dimension of $A(m, n, 2)$

In this section we shall prove Theorem A (Theorem 3.1 below) which establishes a lower bound of three for the CAT(0) dimension of a 3-generator Artin group $A(m, n, 2)$ where $m, n \geq 3$ are both odd. It is known [8] that a 3-generator Artin group $A(m, n, p)$ has geometric dimension 2 for all but the finite number of cases $1/m + 1/n + 1/p > 1$ where the geometric dimension is already 3. (These are precisely the finite type 3-generator Artin groups). These results combined with Theorem 4.2 yield the result stated in the title of the paper.

Theorem 3.1. *Let G be a 3-generator Artin group of type $(m, n, 2)$ where m, n are odd integers (≥ 3). Then G does not act properly discontinuously by semi-simple isometries on a CAT(0) space of covering dimension 2. In particular, the CAT(0) dimension of G is at least 3, and G is not the fundamental group of any compact, non-positively curved 2-complex.*

Remark. (1) As a result of Theorem 3.1, we see that any Artin group which contains a special subgroup $A(m,n,2)$ with m,n odd, has CAT(0) dimension at least 3, since the restriction of a semi-simple action to a subgroup always gives a semi-simple action.

(2) It may be possible to eliminate the hypothesis that m,n be odd. Its use here is largely confined to the latter part of the proof (Subsection 4.2, Case II).

Proof. We suppose, by way of contradiction, that the Artin group

$$A(m,n,2) \;=\; \langle\, a,b,c \mid (a,b)_m = (b,a)_m,\, (b,c)_n = (c,b)_n,\, [a,c] = 1 \,\rangle\,,\;\; m,n \text{ odd},$$

acts properly discontinuously by semi-simple isometries on a CAT(0) space X of covering dimension 2. We shall assume throughout that $\frac{1}{m} + \frac{1}{n} + \frac{1}{2} \leq 1$ since otherwise $A(m,n,2)$ is a finite type Artin group and has geometric dimension 3. In particular, we have $m,n \geq 3$.

By Theorem 1.3 and the fact that X is 2-dimensional, each $\mathbb{Z} \times \mathbb{Z}$ subgroup of $A(m,n,2)$ determines a unique isometrically embedded flat plane in X. We use the notation $\Pi(x,y)$ for the flat plane $Min(x) \cap Min(y)$ associated to a pair of commuting elements x and y. By a well-known result of H. van der Lek [11], the pairs of generators $\{a,b\}$, $\{b,c\}$ and $\{a,c\}$ generate the groups $A(m)$, $A(n)$ and $A(2)$ respectively as subgroups of $A(m,n,2)$. This gives rise to various $\mathbb{Z} \times \mathbb{Z}$ subgroups of $A(m,n,2)$. Our proof of Theorem 3.1 involves analyzing the configuration of the five flat planes

$$\Pi(a, z_{a,b})\,, \quad \Pi(b, z_{a,b})\,, \quad \Pi(b, z_{b,c})\,, \quad \Pi(c, z_{b,c})\,, \quad \Pi(a,c)$$

corresponding (respectively) to the following rank two free abelian subgroups of $A(m,n,2)$

$$\langle\, a,\, z_{a,b} \,\rangle\,, \quad \langle\, b,\, z_{a,b} \,\rangle\,, \quad \langle\, b,\, z_{b,c} \,\rangle\,, \quad \langle\, c,\, z_{b,c} \,\rangle\,, \quad \langle\, a,\, c \,\rangle\,.$$

Here the $z_{s,t}$ denotes the central element in the 2-generator Artin subgroup generated by s and t. Note that $z_{a,c} = ac = ca$ and that $\Pi(a, z_{a,c}) = \Pi(c, z_{a,c}) = \Pi(a,c)$. At this point we make no assumption that the five planes just introduced are mutually distinct (even though there may be quite valid group theoretic reasons why they should be).

The first problem we encounter is that we have *a priori* no picture of how all five planes can intersect each other in X. So we begin by understanding how certain pairs of planes arrange themselves in X. We then move from this local configuration (pairs of planes) to the general configuration of all five planes.

We begin by considering those pairs of planes corresponding to abelian subgroups which share a common central element $z_{r,s}$. For each pair $(r,s) \in \{(a,b),(b,c)\}$, Proposition 2.2 ensures that the periodic flat planes $\Pi(r, z_{r,s})$ and $\Pi(s, z_{r,s})$, defined above, intersect in the nonempty set $B(r,s) = Min(r) \cap Min(z_{r,s}) \cap Min(s)$. Let

$$\Sigma(r,s) \;=\; \Pi(r, z_{r,s}) \cup \Pi(s, z_{r,s})$$

denote the union of these two flat planes in X, and which is in fact a convex subset of X (see Lemma 1.4). We also write $\Sigma(a,c) = \Pi(a,c)$, for consistency.

Next we consider those pairs of planes which correspond to abelian subgroups which have an Artin generator in common. For the pair of planes $\Pi(a, z_{a,b})$ and $\Pi(a, z_{a,c})$ we define $H(a)$ to be the minimal convex subspace of X which contains their union as in Lemma 1.4. In what follows we will use the terminology introduced in Lemma 1.4 to distinguish the two cases where these planes are disjoint ($H(a)$ is H-type) or intersect ($H(a)$ is X-type or B-type, or the two planes coincide). We similarly define $H(b)$ and $H(c)$.

We illustrate in Figure 2 the possible links $\mathrm{Lk}(v, \Sigma(r,s))$ and $\mathrm{Lk}(v, H(s))$ for a point $v \in \Sigma(r,s) \cap H(s)$. The angles $\rho = d(r^+, z^+_{r,s})$ and $\sigma = d(s^+, z^+_{r,s})$ are indicated in the figure. The only constraint on these angles is that $\rho + \sigma < \pi$. This comes from part (3) of Proposition 2.2. (At this point we are using the hypothesis that m, n are odd, and in fact $\rho = \sigma$).

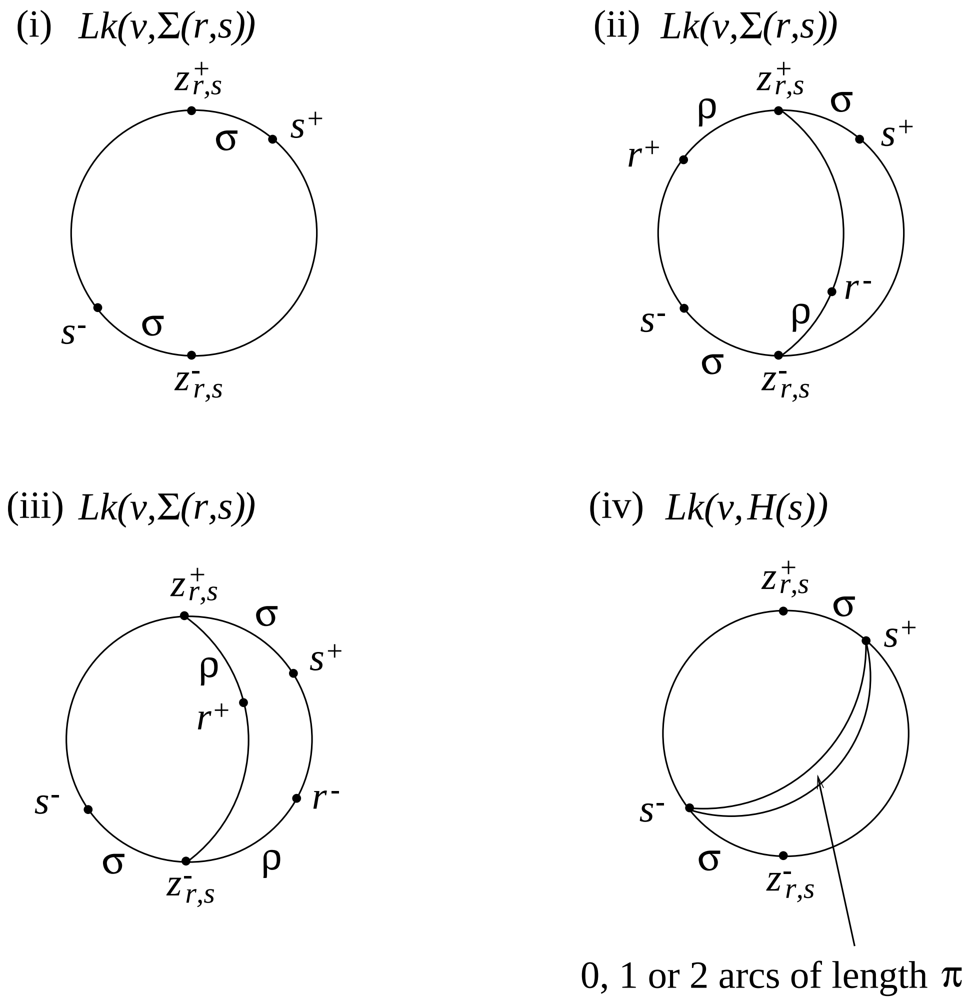

Figure 2: Links of a point in $\Sigma(r,s)$ and in $H(s)$.

The union of the $\Sigma(r,s)$'s and $H(r)$'s forms a (piecewise Euclidean) subspace of X which may be viewed as an identification space of the disjoint union Θ of $\Sigma(a,b)$, $\Sigma(b,c)$, $\Pi(a,c)$, $H(a)$, $H(b)$ and $H(c)$. However, it is not immediately clear exactly what identifications are involved in the map $\Theta \to X$. We do know, for example, that a plane $\Pi(a, z_{a,b})$ in $\Sigma(a,b)$ is identified with a plane in $H(a)$, and so on. But there may, a priori, be further identification. We let Y denote the intermediate identification space which results from those isometric identifications just mentioned. Namely, for each of the five planes $\Pi(r, z_{r,s})$ one has two isometric embeddings, $p : \Pi(r, z_{r,s}) \to H(r) \to \Theta$ and $q : \Pi(r, z_{r,s}) \to \Sigma(r, s) \to \Theta$, and Y is obtained from Θ by identifying the pairs of points $\{p(x), q(x)\}$ for all $x \in \Pi(r, z_{r,s})$ and all $r, s \in \{a, b, c\}$.

The space Y may alternatively be described as the union of $H(a), H(b)$ and $H(c)$ identified along the subsets $B(a, b)$ (common to $H(a)$ and $H(b)$), $B(b, c)$ and $B(a, c) = \Pi(a, c)$.

There is a canonical map $\psi : Y \to X$ induced by the inclusion of each of $H(a)$, $H(b)$ and $H(c)$ into X.

The proof of Theorem 3.1 breaks into two cases. In the first case $H(a)$ is H-type. The case where $H(c)$ is H-type is treated similarly. We obtain a contradiction in these cases by producing a closed geodesic γ in Y whose image $\delta = \psi(\gamma)$ is either a geodesic bigon or a geodesic triangle with angle sum in excess of π in the CAT(0) space X. In the second case neither $H(a)$ nor $H(c)$ are H-type (so both planes $\Pi(a, z_{a,b})$ and $\Pi(c, z_{b,c})$ intersect $\Pi(a, c)$). In this case we can use the structure of minsets developed in Propositions 1.2 and 2.2 to find a fixed point for an infinite order element of $A(m, n, 2)$. The details of these two cases are provided below. This concludes the proof of Theorem 3.1. □

3.1 Case I : $H(a)$ of H-type.

(A similar proof applies in the case that $H(c)$ is H-type).

In this case $\Pi(a, c)$ and $\Pi(a, z_{a,b})$ are joined by an infinite strip $[-1, 1] \times \mathbb{R}$ with each line $\{\pm 1\} \times \mathbb{R}$ identified to an a-invariant line in one of the two planes, forming $H(a)$. Let μ denote the centre line $\{0\} \times \mathbb{R}$ of the strip and express $H(a)$ as $T \cup_\mu T'$ where $T = \Pi(a, z_{a,b}) \cup ([0, 1] \times \mathbb{R})$ and $T' = \Pi(a, z_{a,c}) \cup ([-1, 0] \times \mathbb{R})$. Let μ, μ' denote the preimages of μ in T, T' respectively, and let $h : \mu \to \mu'$ denote the identifying homeomorphism.

Now, cutting Y along μ, we obtain a piecewise Euclidean space

$$Z = T \cup_{B(a,b)} H(b) \cup_{B(b,c)} H(c) \cup_{B(c,a)} T' ,$$

such that $Y = Z/(x \sim h(x),\ \text{for all } x \in \mu)$.

Lemma 3.2. *The space Z, and hence also Y, is locally CAT(0). Moreover, Z is contractible.*

Proof. We use the link condition. Every point in Z has a link which is may be built up, in a finite number of steps, from a 2π-circle by attaching at each step an arc of length π between antipodal points on a circle of length 2π in the link obtained at the previous step. (See Figure 2). The result of such a process will always be a CAT(1) link. Thus Z is locally CAT(0). Links of points in Y are as for points in Z with the exception of points along μ where the link is always a 2π-circle. So Y is also locally CAT(0). Finally, one may easily construct a homotopy retraction of T onto $B(a, b)$, then of $H(b)$ onto $B(b, c)$, then of $H(c)$ onto $B(c, a)$, then of T' to a point, showing that Z is a contractible space.

□

Note that Z may be thought of as a fundamental domain for the action of $\pi_1(Y) \cong \mathbb{Z}$ on the universal cover $\widetilde{Y}$ of Y. Moreover, by the Cartan-Hadamard Theorem ([7], Theorem II.4.1), both $\widetilde{Y}$ and Z are CAT(0) spaces.

Lemma 3.3. *There exists a closed geodesic γ in Y which lifts to a geodesic path $\widetilde{\gamma} : [0,1] \to Z$ with the following properties.*

(i) $\widetilde{\gamma}(0) \in \mu$ and $\widetilde{\gamma}(1) = h(\widetilde{\gamma}(0)) \in \mu'$,

(ii) there exist $0 \leq t_{a,b} \leq t_{b,c} \leq t_{c,a} \leq 1$ such that

$$\widetilde{\gamma}(t_{a,b}) \in B(a,b)\,, \qquad \widetilde{\gamma}(t_{b,c}) \in B(b,c)\,, \qquad \widetilde{\gamma}(t_{c,a}) \in B(c,a)\,,$$

and

$$\widetilde{\gamma}([0,t_{a,b}]) \in T\,, \qquad \widetilde{\gamma}([t_{a,b},t_{b,c}]) \in H(b)\,, \qquad \widetilde{\gamma}([t_{b,c},t_{c,a}]) \in H(c)\,, \qquad \widetilde{\gamma}([t_{c,a},1]) \in T'\,.$$

Proof. The space Y may be triangulated, and hence $\widetilde{Y}$ equivariantly triangulated, into a piecewise Euclidean complex with finitely many isometry types of simplexes. Thus, by [7], II.6.6 Exercise (2), every simplicial isometry of $\widetilde{Y}$ is semi-simple. In particular, the deck transformation $g : \widetilde{Y} \to \widetilde{Y}$ associated to a generator of $\pi_1(Y) \cong \mathbb{Z}$ is a semi-simple with $\ell(g) \neq 0$. Thus $Min(g)$ contains a g-axis whose image in Y is a closed geodesic γ with the desired properties. For (i), note that $h = g\,|_{\mu}$. Part (ii) follows from the definition of Z and is true for any geodesic from a point on μ to a point on μ'. We note that T, $H(b)$, $H(c)$ and T' are each closed convex subsets of Z.

□

We now consider the closed geodesic $\gamma : S^1 \to Y$ of Lemma 3.3, where S^1 is decomposed into a union of closed intervals $I_a = [t_{c,a}, 1] \cup [0, t_{a,b}]$, $I_b = [t_{a,b}, t_{b,c}]$ and $I_c = [t_{b,c}, t_{c,a}]$. Let δ denote the closed path in X obtained by composing γ with the map $\psi : Y \to X$. For each $s \in \{a,b,c\}$, let δ_s denote the path obtained by restricting δ to the interval I_s. Note that the restriction of γ to I_s is a locally geodesic path in Y with image in $H(s)$ and hence defines a geodesic in the CAT(0) space $H(s)$. Since each of $H(a), H(b)$ and $H(c)$ is a convex subspace of X, it now follows that each of δ_a, δ_b and δ_c is a geodesic path in X. Thus δ is a (possibly degenerate) geodesic triangle in X with sides $\delta_a, \delta_b, \delta_c$ and corners $\delta(t_{a,b}), \delta(t_{b,c}), \delta(t_{c,a})$.

For $r, s \in \{a,b,c\}$, we say that $t_{r,s}$ is a *non-degenerate corner point* of δ if both δ_r and δ_s are sides of nonzero length.

Note that at least δ_a is a side of nonzero length (it must cross the strip $[-1,1] \times \mathbb{R}$ in $H(a)$). Thus we cannot have $t_{a,b} = t_{b,c} = t_{c,a}$ for then δ_a would be a geodesic in X of nonzero length which starts and finishes at the same point, a contradiction.

Therefore, we are in one of the following cases:

1. δ is a non-degenerate triangle ($t_{a,b}$, $t_{b,c}$ and $t_{c,a}$ are distinct non-degenerate corner points),

2. δ is a bigon with one non-degenerate corner point whose opposite side has zero length.

But it follows from Lemma 3.5 below that we now have, in case (1), a geodesic triangle with angle sum greater than π, and in case (2), a bigon with at least one nonzero angle. Both of these outcomes contradict X being a CAT(0) space, which completes the reduction of our proof to Case II.

Before proving Lemma 3.5 we need to establish the following Lemma.

Lemma 3.4. *Suppose that $f : U \to V$ is a locally isometric map from a geodesic space U to a CAT(1) space V. Suppose that $p, q \in U$ are such that $d_U(p,q) \geq \pi$. Then*

$$d_V(f(p), f(q)) \geq 2\pi - d_U(p,q)\,.$$

Proof. Let $[p,q]$ denote a geodesic segment in U from p to q, and let r be the midpoint of $[p,q]$. We may as well suppose that $d_U(p,q) < 2\pi$, in which case $[p,q]$ is the union of geodesic segments $[p,r]$ and $[r,q]$ each of length $\frac{1}{2}d_U(p,q)$ which is less than π. Since f is locally isometric and V is CAT(1), these map to (unique) geodesic segments $[f(p), f(r)] = f([p,r])$ and $[f(r), f(q)] = f([r,q])$ in V. Moreover, since f is a local isometry at the point r, and $[p,q]$ is geodesic, we also have that the Alexandrov angle at $f(r)$ between these geodesic segments is π.

Now suppose, by way of contradiction, that $d_V(f(p), f(q)) < 2\pi - d_U(p,q)$. Then, since $d_U(p,q) > \pi$, we have $d_V(f(p), f(q)) < \pi$ and therefore a unique geodesic segment $[f(p), f(q)]$ in V. We now have a triangle $\Delta = \Delta(f(p), f(q), f(r))$ in V with perimeter $d_V(f(p), f(q)) + d_U(p,q) < 2\pi$. But then any comparison triangle in $\mathbb{S}^2$ for Δ has all angles strictly less than π. Thus Δ, having an angle of π, fails the CAT(1) inequality ([7], Proposition II.1.7(4)), a contradiction. □

Lemma 3.5. *Suppose that, for some $r, s \in \{a,b,c\}$, the point $t_{r,s}$ is a non-degenerate corner point of δ. Then the geodesic paths δ_r and δ_s form an angle $\theta_{r,s}$ at $\delta(t_{r,s})$ in X where $\theta_{a,c} \geq \frac{\pi}{2}$ and $\theta_{a,b}, \theta_{b,c} \geq \frac{\pi}{3}$.*

Proof. Let $v = \delta(t_{r,s})$. Then v lies in each of the closed convex subspaces $H(r)$, $H(s)$ and $\Sigma(r,s)$ of X. We now define the metric graph L to be the identification space

$$L = \mathrm{Lk}(v, H(r)) \cup \mathrm{Lk}(v, \Sigma(r,s)) \cup \mathrm{Lk}(v, H(s)) \,/\sim$$

where $\mathrm{Lk}(v, H(r))$ and $\mathrm{Lk}(v, \Sigma(r,s))$ are identified along the common circular link observed for v in $\Pi(r, z_{r,s})$, and $\mathrm{Lk}(v, H(s))$, $\mathrm{Lk}(v, \Sigma(r,s))$ are identified along the circular link of v in $\Pi(s, z_{r,s})$. Note that L is a geodesic space, and may be viewed as a sub-link of the link $\mathrm{Lk}(\gamma(t_{r,s}), Y)$ where γ is the closed geodesic of Lemma 3.3.

We now make use of the space of directions $S_v(X)$ which may be defined for any point v in a metric space X. A theorem of Nikolaev [12] (see also [7]) states that if X is a CAT(0) space then the metric completion $S_v(X)'$ of $S_v(X)$ is a CAT(1) space. (Any space is isometrically embedded in its metric completion).

The inclusions of each subspace into X together induce a map $f : L \to S_v(X) \subset S_v(X)'$, which we observe to be locally isometric as follows. A neighbourhood of any point of L lies wholly in the subgraph $\mathrm{Lk}(v, R)$ where the subspace R is one of $H(r), H(s)$ or $\Sigma(r,s)$. (For this assertion we are using the fact that $\rho + \sigma < \pi$ so that r^+ does not coincide with s^- or s^+ with r^- in L. See Figure 3 for the possible forms of L.) Since each of these subspaces R embeds in X as a convex subspace, f restricts to a locally isometric embedding on $\mathrm{Lk}(v, R)$. Hence any point in L possesses a neighbourhood which is embedded isometrically in $S_v(X)'$ by f.

We wish to measure the Alexandrov angle between δ_r and δ_s at v. Let $x \in \mathrm{Lk}(v, H(r))$ and $y \in \mathrm{Lk}(v, H(s))$ be such that δ_r determines the point $f(x) \in S_v(X)$ and δ_s determines the point $f(y) \in S_v(X)$. Then the two sides δ_r and δ_s form an angle at v given by $\theta_{r,s} = d^{(\pi)}_{S_v(X)}(f(x), f(y))$.

Note that $d_L(x,y) \geq \pi$, since the fact that γ is a geodesic implies that $d^{(\pi)}(x,y) = \pi$ when viewed in $\mathrm{Lk}(\gamma(t_{r,s}), Y)$. By Lemma 3.4 we now have $\theta_{r,s} \geq 2\pi - d_L(x,y)$. The result now follows by Lemma 3.6 which shows that $d_L(x,y) \leq \frac{5\pi}{3}$, and $\leq \frac{3\pi}{2}$ in the case $m_{r,s} = 2$. □

Lemma 3.6. *Let L be the metric graph obtained from $Lk(v, H(r))$, $Lk(v, \Sigma(r,s))$ and $Lk(v, H(s))$ as in the proof of Lemma 3.5. Define $diam(L) = sup\{d_L(u,v) : u, v \in L\}$.*

(1) If $m_{r,s} \geq 3$, then $diam(L) \leq \frac{5\pi}{3}$, and

(2) if $m_{r,s} = 2$, then $diam(L) \leq \frac{3\pi}{2}$.

Proof. In view of the possibilities illustrated in Figure 2, L takes the form of one of the links shown in Figure 3(i) if $m_{r,s} = 2$, that is if $\{r,s\} = \{a,c\}$, and otherwise any of Figures 3(i) or 3(ii), or 3(ii) with the roles of r and s reversed.

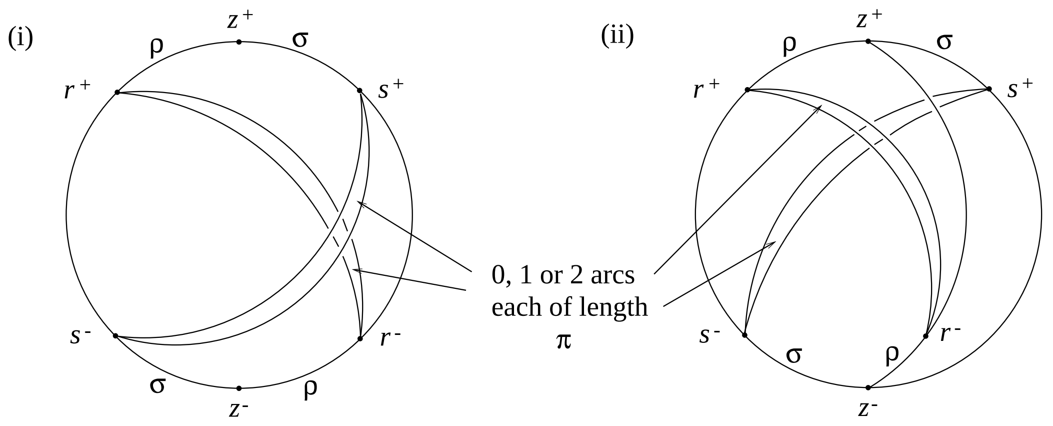

Figure 3: Possible forms of the metric graph L.

Each of the links illustrated in Figure 3 consists of $\mathrm{Lk}(v, \Sigma(r,s))$ with zero, one or two arcs of length π connecting between r^+ and r^-, and similarly between s^+ and s^-. We shall call these additional arcs $H(r)$-arcs and $H(s)$-arcs respectively.

We consider first the graphs illustrated in Figure 3(i). This includes the case where $m_{r,s} = 2$. In fact if a graph L is formed from a 2π circle by adding arcs of length π onto at most two 'great 0-circles' (pairs of antipodal points), then the diameter of L is at most $\pi + \phi$ where ϕ denotes the distance between the two great 0-circles. One can easily see this by producing a family $\{c_i\}$ of closed geodesics of length either 2π or $2(\pi + \phi)$ such that every pair of points in L lie on a common c_i. In particular $\mathrm{diam}(L) \leq \frac{3\pi}{2}$.

Now consider the links illustrated in Figure 3(ii). Take any $p, q \in L$. Note that by the reasoning just given, the union of $\mathrm{Lk}(v, \Sigma(r,s))$ with just the $H(r)$-arcs (or with just the $H(s)$-arcs) has diameter at most $\frac{3\pi}{2}$. Thus we may as well assume that p lies in an $H(r)$-arc and q lies on an $H(s)$-arc.

Now consider the distances in $\mathrm{Lk}(v, \Sigma(r,s))$ between either of the points $r^\pm$ and either of $s^\pm$ which may be displayed in a matrix as follows:

$$\begin{pmatrix} d(r^+,s^+) & d(r^+,s^-) \\ d(r^-,s^+) & d(r^-,s^-) \end{pmatrix} = \begin{pmatrix} \rho+\sigma & \pi-(\rho+\sigma) \\ \pi-|\rho-\sigma| & \rho+\sigma \end{pmatrix}.$$

We describe four locally geodesic paths from p to q which we denote $\tau(\epsilon,\nu)$, for $\epsilon,\nu \in \{+,-\}$ as follows. The path $\tau(\epsilon,\nu)$ passes through r^ϵ and s^ν and follows in part the unique geodesic in $\mathrm{Lk}(v,\Sigma(r,s))$ of length $d(r^\epsilon,s^\nu)$. The paths $\tau(+,+)$ and $\tau(-,-)$ together form a closed geodesic in L of total length $2\pi + d(r^+,s^+) + d(r^-,s^-)$ which is an upper bound for $2d(p,q)$. Similarly with $\tau(+,-)$ and $\tau(-,+)$. This gives the following upper bound for $d(p,q)$ (which is, in fact, precisely the diameter of L):

$$\begin{aligned} d(p,q) &\leq \pi + \frac{1}{2}\min\{d(r^+,s^+)+d(r^-,s^-), d(r^+,s^-)+d(r^-,s^+)\} \\ &= \pi + \min\{\rho+\sigma, \pi - \frac{1}{2}(\rho+\sigma+|\rho-\sigma|)\} \\ &= \pi + \min\{\rho+\sigma, \pi-\max\{\rho,\sigma\}\} \\ &= \pi + \min\{\rho+\sigma, \pi-\rho, \pi-\sigma\}. \end{aligned}$$

This upper bound is greatest when $\rho=\sigma=\frac{\pi}{3}$. Thus we always have $\mathrm{diam}(L) \leq \frac{5\pi}{3}$.

□

3.2 Case II : Both $\Pi(a, z_{a,b})$ and $\Pi(c, z_{b,c})$ intersect $\Pi(a,c)$.

We obtain a contradiction in this case by proving that the set

$$R := Min(a) \cap Min(b) \cap Min(c) = Min(b) \cap \Pi(a,c)$$

contains a point which is fixed by an infinite order group element. In order to give a more concrete description we first consider the following subsets of R. Define

$$R(a,b) := R \cap Min(z_{a,b}) = R \cap \Pi(b, z_{a,b}) \text{ and,}$$

$$R(b,c) := R \cap Min(z_{b,c}) = R \cap \Pi(b, z_{b,c}).$$

Note that $R(a,b) = B(a,b) \cap \Pi(a,c)$. In fact, if we let $S(a)$ denote the nonempty subset $\Pi(a,z_{a,b}) \cap \Pi(a,c)$ of $H(a)$, then $R(a,b)$ should be thought of as the part of $B(a,b)$ which intersects $S(a)$. Now $S(a)$ is a union of a-axes: either a single a-axis ($H(a)$ is X-type), an interval's worth $I \times \mathbb{R}$ ($H(a)$ is B-type), or the whole of $\Pi(a, z_{a,b})$. Each a-axis (of $\Pi(a,z_{a,b})$) intersects $B(a,b)$ transversely in a closed interval, which we call an *a-segment*. The set $R(a,b)$ is then a closed convex union of (parallel) a-segments: either a single a-segment, a parallelogram bounded on two sides by a-segments and on the other sides by the sides of $B(a,b)$, or just the whole of $B(a,b)$ (see Figure 4).

We have similar possible descriptions for the set $R(b,c)$ as a union of c-segments lying in the band $B(b,c)$. We now have the following description of R:

Lemma 3.7. *The set R is a 2-dimensional closed convex subset of $\Pi(a,c)$ and $R = R(a,b) = R(b,c)$.*

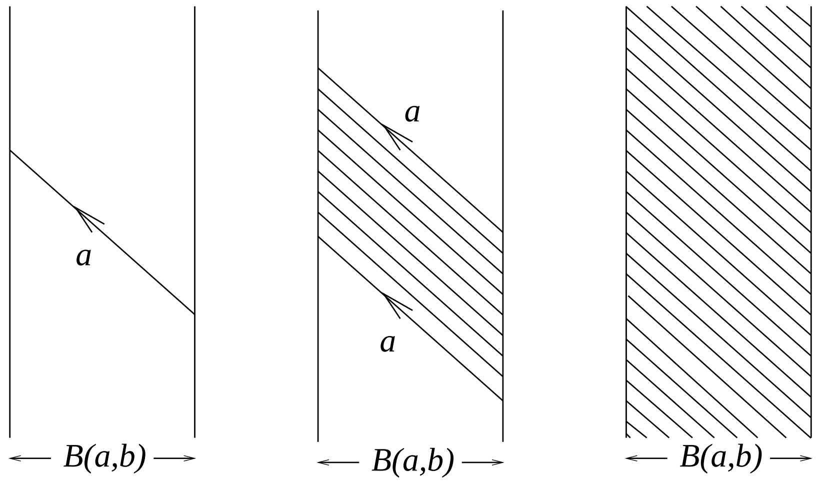

Figure 4: $R(a,b)$ as a subset of $B(a,b)$

Proof. Clearly, as an intersection of minsets, R is a closed convex subset of $\Pi(a,c)$ which contains both $R(a,b)$ and $R(b,c)$. In particular R contains the convex hull of the union of an a-segment from $R(a,b)$ and a c-segment from $R(b,c)$. Since these segments lie in different directions in the plane $\Pi(a,c)$, it follows that R is a 2-dimensional (possibly infinite) closed convex subset of $\Pi(a,c)$.

If $R \neq R(a,b)$ then one can find a point x of $R(a,b)$ which lies in the closure of $R \setminus R(a,b)$. Thus $\mathrm{Lk}(x,R)$, which is closed connected subset of $\mathbb{S}^1$, contains $\mathrm{Lk}(x,R(a,b))$ as a proper closed subset. In particular, $\mathrm{Lk}(x,R)$ contains a nontrivial closed arc which intersects $\mathrm{Lk}(x,R(a,b))$ at a single point, necessarily one of $a^{\pm}$ or $z^{\pm}_{a,b}$.

Note that $R \cup \Pi(b, z_{a,b})$ is a convex subset of $Min(b)$ where R intersects $\Pi(b, z_{a,b})$ along the set $R(a,b)$. Considering the contributions from R and $\Pi(b, z_{a,b})$, we see that $S_x(Min(b))$ contains a tripod graph whose valence three vertex is one of $a^{\pm}$ or $z^{\pm}_{a,b}$. But this contradicts Proposition 1.2(2) since, by Proposition 2.2 (1) and (3), b-axes are transverse to both a-axes and to $z_{a,b}$-axes. Thus $R = R(a,b)$. Similarly, one can show that $R = R(b,c)$. $\square$

There are now only two possibilities for the region R. Either $B(a,b)$ and $B(b,c)$ coincide and $R = B(a,b) = B(b,c)$, or R is a nondegenerate parallelogram bounded by a pair of a-segments and a pair of c-segments. We treat the cases separately. In both cases we apply Proposition 2.2 (2) to each of $B(a,b)$ and $B(b,c)$, thus using the fact that both m and n are odd.

(i) $B(a,b)$ and $B(b,c)$ coincide in R.

In this case the ba-invariant edge of $B(a,b)$, which we shall denote ℓ, must coincide with the bc-invariant edge of $B(b,c)$. Moreover, since $ba(\ell) = bc(\ell) = \ell$, we have $a(\ell) = c(\ell) = b^{-1}(\ell)$, another line in R parallel to ℓ. By considering translation lengths in the direction of these lines one now sees that $abc(p) = cba(p)$, for all $p \in \ell$. (See Figure 5(i)). Thus every point of ℓ is fixed by the element $g = a^{-1}b^{-1}c^{-1}abc \in A(m,n,2)$. There is a canonical map of $A(m,n,2)$ onto the Coxeter group

$$W(m,n,2) = \langle a,b,c \mid a^2 = b^2 = c^2 = 1, (ab)^m = (bc)^n = (ac)^2 = 1 \rangle$$

which takes g to the element $(abc)^2$. When $1/m+1/n+1/2 \leq 1$ the group $W(m,n,2)$ is isomorphic to the group generated by reflections in the sides of a Euclidean or hyperbolic triangle with angles π/m, π/n and $\pi/2$, and the element abc is known to be an element of infinite order in this group. Therefore g must be an element of infinite order, giving our contradiction.

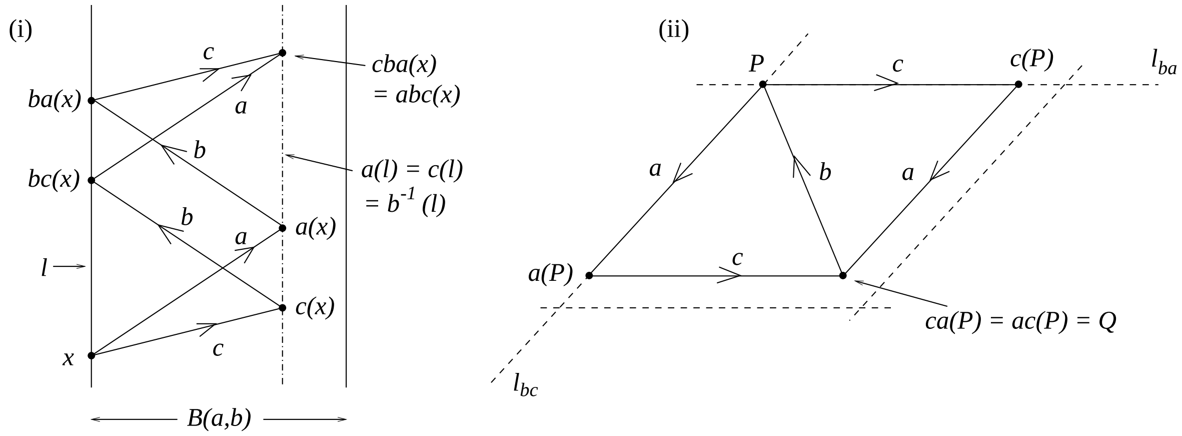

Figure 5: Finding a fixed point in R

(ii) R is a parallelogram.

We use the fact that the lengths of the a-segments and c-segments which bound R in this case are respectively greater than or equal to the translation lengths $\ell(a)$ and $\ell(c)$. Thus there is a (unique) vertex P of R for which both $a(P)$ and $c(P)$ lie in R. Let $Q = ac(P) = ca(P)$. Using the fact that $R = R(a,b)$ and recalling Proposition 2.2, we observe that the interval segment $[P, c(P)]$ lies on a ba-axis ℓ_{ba}. Similarly, by viewing R as $R(b,c)$, the segment $[P, a(P)]$ lies on a bc-axis ℓ_{bc}.

Note that the segment $[P, a(P)]$ (which lies on an a-axis) meets ℓ_{ba} transversely when viewed in the flat plane $\Pi(a, z_{a,b})$. It follows that ℓ_{ba} and ℓ_{bc} intersect transversely at P, and in particular that $\ell_{ba} \cap \ell_{bc} = \{P\}$.

Now, since $c(P) \in \ell_{ba}$ and ℓ_{ba} is ba-invariant, it follows that $b(Q) = bac(P)$ also lies in ℓ_{ba}. But, by the same reasoning, $b(Q)$ lies in ℓ_{bc}. Therefore we must have $b(Q) = P$. Now it follows that abc fixes the point $a(P)$ (see Figure 5(ii)) and, as was observed above, abc is an element of infinite order in $A(m,n,2)$. Thus we again have a contradiction.

This completes the proof of Case II, and of Theorem 3.1.

4 Artin groups with 3-dimensional non-positive curvature

In this section we produce 3-dimensional CAT(0) structures for a large class of Artin groups, including all but finitely many of the 3-generator Artin groups, thus establishing Theorem B of the Introduction (see Theorem 4.2 and Corollary 4.3 below).

4.1 The building blocks

We begin by constructing a 3-dimensional locally CAT(0) Eilenberg-Mac Lane complex X_m for each dihedral type Artin group

$$A(m) \;=\; \langle\, a, b \mid (a,b)_m = (b,a)_m \,\rangle \text{ , for } m \geq 3.$$

This complex, at least for the trefoil knot group $A(3)$, was previously known to a number of researchers in the field. It was described to us by B.H. Bowditch who attributed it to Thurston. A description of it is also given by Bestvina [1].

Start with a regular euclidean m-gon, M, centered on the origin in the xy-plane. We shall first describe X_m as a certain quotient space of the infinite m-gon prism, $M \times \mathbb{R} \subset \mathbb{R}^3$. Let $\phi : M \times \mathbb{R} \to M \times \mathbb{R}$ be the homeomorphism which translates a positive distance vertically (in z-direction) and rotates through $(2\pi/m)$ about the z-axis. Let E denote the union of the vertical edges of $M \times \mathbb{R}$; that is, the union of edges of the form $\{v\} \times \mathbb{R}$ where v is a vertex of M. The 3-complex, X_m, is defined to be the quotient of $M \times \mathbb{R}$ by the maps $\phi|_E$ and ϕ^2.

The universal cover of X_m consists of an infinite collection of copies of $M \times \mathbb{R}$ identified in pairs along their edges. A fundamental domain for the action is the regular polygonal prism $M \times I$ (I a compact interval) illustrated in Figure 6. We recover X_m from this fundamental domain as follows. First identify the top and bottom m-gons by a $2(2\pi/m)$-twist to obtain a solid torus N. (This corresponds to the quotient by ϕ^2 above.) The vertical edges (of the form $\{v\} \times I$ where v is a vertex of the m-gon) combine to give curves on the boundary of N : a single circle ξ_0 in the case m odd, and a pair of parallel circles ξ_1, ξ_2 if m is even. The identification of these curves by $\phi|_E$ then either glues ξ_0 to itself by the antipodal map (m odd) or glues ξ_1 to ξ_2 (m even). The image of these curves shall be written ξ in each case. The result of these identifications is X_m.

Note that X_m inherits from $M \times \mathbb{R}$ a natural piecewise Euclidean metric, but that this metric depends on the two factors: the area of M and the translation length of ϕ.

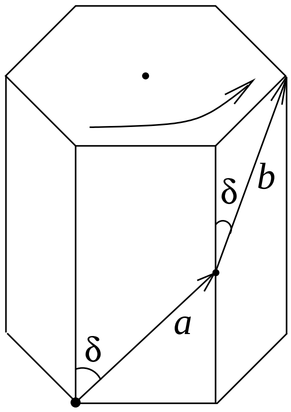

Figure 6: The 3-complex for the dihedral type Artin group $A(m)$.

We choose a basepoint v in X_m and indicate, in Figure 6, a pair of closed geodesics in X_m, labeled a and b, which pass through v. (These appear as geodesic paths in the universal cover, or in the prism $M \times I$.) The precise metric on X_m shall now be chosen so that a and b both have length 1, and then depends only on the acute angle δ between each of these closed geodesics and the vertical lines (see Figure 6). We write $X_m(\delta)$ when we wish to make the metric explicit. Note that the metric may be chosen so that δ takes any value in the range $0 < \delta < \frac{\pi}{2}$.

The closed geodesics a and b will be called *generator curves.* They represent generators a and b for the fundamental group so that $\pi_1(X_m, v) = A(m; a, b)$. This may be seen by applying Van Kampen's Theorem. The element $x = ab$ is represented by a curve running once around the core of the solid torus N, and $\Delta = (a, b)_m$ is represented by the curve ξ. Observe that

$$[\xi_0] = x^m = \Delta^2 = z \quad \text{if } m \text{ odd, and}$$

$$[\xi_1] = x^{m/2} = \Delta = z \quad \text{if } m \text{ even,}$$

where $z = z_{a,b}$ generates the centre of $A(m)$.

In Figure 7 we see the link of v in X_m. It is a piecewise spherical 2-complex which can be described as the orthogonal join of a 0-sphere with the union of two geodesic segments of length $\theta = \frac{(m-2)\pi}{m}$. We have indicated the points $a^{\pm}$ and $b^{\pm}$ corresponding to the geodesic loops based at v which represent the generators a and b respectively. The 0-sphere points correspond to the geodesic loop ξ (based at v) which runs in the direction of the the central element $z_{a,b}$.

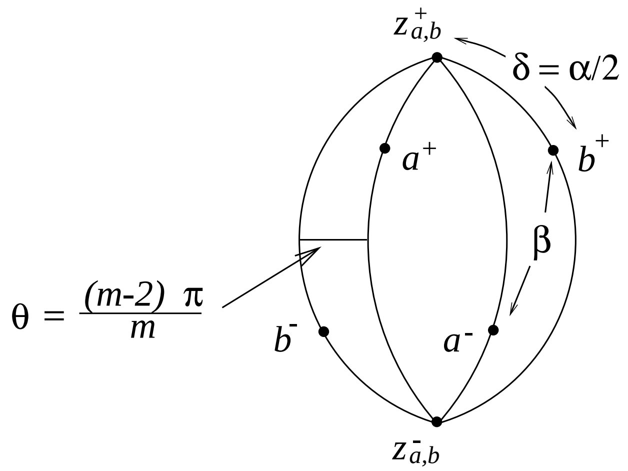

Figure 7: The link of the vertex v in the 3-complex X_m.

It is easy to see, by checking the link condition, that X_m is locally CAT(0). Its universal cover is a CAT(0) space which decomposes as a metric product of $\mathbb{R}$ with a 2-complex which is composed of infinitely many regular euclidean m-gons attached together along their vertices in a tree-like pattern. Each m-gon is attached to m others, one along each of its m vertices. Note that this 2-complex can be $A(m)$-equivariantly retracted down onto a Bass-Serre tree for $A(m)/Z = \mathbb{Z}_m * \mathbb{Z}_2$. In fact one can see in this way that X_m has, as a 2-spine, an equivariantly embedded copy of the CAT(0) 2-complex described by T. Brady and J. McCammond in [4].

4.2 Combining the building blocks

We shall use the complexes $X_m(\delta)$, $0 < \delta < \frac{\pi}{2}$, as building blocks for constructing complexes for Artin groups with three or more generators. We adopt the convention that $X_2(\delta)$ is a 2-torus where the angle between geodesic representatives of the two fundamental group generators is 2δ.

Note that in the link of the vertex v in the complex $X_m(\delta)$, as shown in Figure 7, the points $a^{\pm}$ and $b^{\pm}$ are all a distance δ from the zero sphere $\{z^{\pm}_{a,b}\}$.

Define angles α and β by

$$\alpha \;=\; d(a^+, b^+) \;=\; d(a^-, b^-) \;=\; 2\delta$$

and

$$\beta \;=\; d(a^+, b^-) \;=\; d(a^-, b^+)\,.$$

When we come to combining the $X_m(\delta)$ spaces by gluing them along the a and b generator curves, the resulting complexes will have vertex links obtained by combining several of the links in Figure 7 together along the $\{a^{\pm}\}$ and $\{b^{\pm}\}$ spheres. The key geometric observation in this section is that for large values of m one can arrange that both α and β are large enough that the new links one obtains by such combinations will still be CAT(1).

First we need a lemma describing the precise relationship between α, β and m.

Lemma 4.1. *Let $m \geq 3$ be an integer, and let $\theta = \frac{(m-2)\pi}{m}$. Then α, β and θ are related by the equation*

$$2\cos\beta \;+\; (1+\cos\theta)\cos\alpha \;+\; (1-\cos\theta) \;=\; 0\,.$$

Proof. We have $d(a^-, z^-_{a,b}) \;=\; \delta \;=\; \alpha/2 \;=\; d(b^+, z^+_{a,b})$, and we wish to compute $\beta \;=\; d(a^-, b^+)$. Realize the half of the link in Figure 7 which contains a^- and b^+ as a subset of the unit sphere in $\mathbb{R}^3$, with $z^+_{a,b}$, $z^-_{a,b}$, a^- and b^+ mapping to the unit vectors $\langle 0,0,1\rangle$, $\langle 0,0,-1\rangle$, $\langle \sin\delta, 0, -\cos\delta\rangle$ and $\langle \sin\delta\cos\theta, \sin\delta\sin\theta, \cos\delta\rangle$ respectively.

Then

$$\begin{aligned}\cos\beta \;&=\; \langle \sin\delta, 0, -\cos\delta\rangle \cdot \langle \sin\delta\cos\theta, \sin\delta\sin\theta, \cos\delta\rangle \\ &=\; \sin^2\delta\cos\theta \;-\; \cos^2\delta \\ &=\; \left(\frac{1-\cos\alpha}{2}\right)\cos\theta \;-\; \left(\frac{1+\cos\alpha}{2}\right)\end{aligned}$$

and the result follows on rearranging terms.

$\square$

For example, in the case $\alpha = \beta$ we can solve the equation in Lemma 4.1 above to get

$$\cos\alpha \;=\; \frac{\cos\theta - 1}{\cos\theta + 3}\,.$$

Table 1 shows how the $\alpha = \beta$ values depend on m. Angle measurements are in degrees. It will be useful in proving Theorem 4.2 below.

Theorem 4.2. *All but finitely many of the 3-generator Artin groups of the form $A(m,n,2)$ with $3 \leq m, n < \infty$ are the fundamental groups of compact locally CAT(0) 3-complexes.*

m	$\theta = \frac{(m-2)\pi}{m}$	$\cos\theta$	$\cos\alpha = \cos\beta$	$\alpha = \beta$
3	60	0.5	-1/7	98.213
4	90	0	-1/3	109.471
5	108	-0.309	-0.486	119.107
6	120	-0.5	-3/5	126.870
7	128.571	-0.623	-0.683	133.090
8	135	-$\sqrt{2}/2$	-0.745	138.118
9	140	-0.766	-0.791	142.237
10	144	-0.809	-0.826	145.656
11	147.273	-0.841	-0.853	148.531
12	150	-0.866	-0.874	150.978
13	152.307	-0.885	-0.892	153.083
18	160	-0.940	-0.941	160.298
19	161.053	-0.946	-0.947	161.306
21	162.857	-0.9556	-0.9565	163.046
22	163.64	-0.9595	-0.9603	163.801
43	171.628	-0.9893	-0.9894	171.650
44	171.818	-0.9898	-0.9899	171.839

Table 1: Dependence of $\alpha = \beta$ values on m.

Proof. We construct a space by gluing together the building blocks for the $A(m)$, $A(n)$ and $A(2)$ as follows. Let

$$X_{m,n,2} = X_m(\delta_1) \cup X_n(\delta_2) \cup X_2(\pi/2)/\sim$$

where we identify all three basepoints to a point v and identify the generator curves from each building block in pairs in the obvious fashion so that the fundamental group of $X_{m,n,2}$ is $A(m,n,2)$. We now compute the values of m and n for which $X_{m,n,2}$ is locally CAT(0).

To apply the link condition we first observe that all the 2-dimensional piecewise spherical links in X are locally CAT(1), and so will be CAT(1) precisely when they have no closed geodesic loops of length less than 2π.

Suppose for concreteness that $A(m,n,2)$ has Artin generators a, b and c where a and b satisfy the relation of length $2m$, b and c satisfy the relation of length $2n$, and a and c commute. The link of the vertex in the complex $X_2(\pi/2)$ for $\langle a, c\rangle$ is a circle with two linking zero spheres labeled by $\{a^+, a^-\}$ and by $\{c^+, c^-\}$. The link of the vertex in the 3-complex for $\langle a, b\rangle$ (and likewise for $\langle b, c\rangle$) is as shown in Figure 7. These links combine together by identifying points with the same label from the set $\{a^+, a^-, b^+, b^-, c^+, c^-\}$. The building block links are all CAT(1), so the only place to look for short loops is when one combines three links as above. We represent the combined link schematically as shown in Figure 8, where we use subscripts to distinguish the (possibly different) values for α and β in each of $X_m(\delta_1)$ and $X_n(\delta_2)$. There are six distances to measure between the points of any two of the three 0-spheres $\{a^+, a^-\}$, $\{b^+, b^-\}$ and $\{c^+, c^-\}$. We record these distances schematically using a complete graph on 4 vertices. The link shown in Figure 8 is the union of three such graphs.

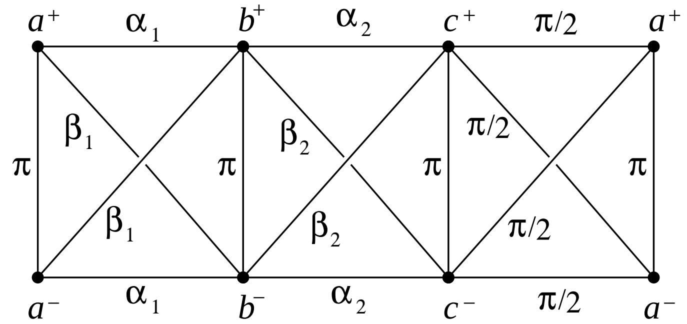

Figure 8: Schematic picture of link of a vertex in the complex for $A(m,n,2)$.

Working with model 3-complexes where $\alpha_i = \beta_i$ it is clear that one just needs to ensure that

$$\alpha_1 + \alpha_2 + \pi/2 \geq 2\pi$$

in order to get combined links which have no loops of length less than 2π. There are a few cases to consider.

Case $A(m,3,2)$. In this case $\alpha_2 = \beta_2 = 98.21$ from Table 1. We see that m must be chosen large enough so that $\alpha_1 = \beta_1$ is greater than $360 - (98.21 + 90) = 171.79$. Table 1 shows that this is achieved for $m \geq 44$.

Case $A(m,4,2)$. In this case $\alpha_2 = \beta_2 = 109.5$ from Table 1. We see that m must be chosen large enough so that $\alpha_1 = \beta_1$ is greater than $360 - (109.5 + 90) = 160.5$. Table 1 shows that this is achieved for $m \geq 19$.

Case $A(m,5,2)$. In this case $\alpha_2 = \beta_2 = 119.1$ from Table 1. We see that m must be chosen large enough so that $\alpha_1 = \beta_1$ is greater than $360 - (119.1 + 90) = 150.9$. Table 1 shows that this is achieved for $m \geq 12$.

Case $A(m,6,2)$. In this case $\alpha_2 = \beta_2 = 126.8$ from Table 1. We see that m must be chosen large enough so that $\alpha_1 = \beta_1$ is greater than $360 - (126.9 + 90) = 143.1$. Table 1 shows that this is achieved for $m \geq 10$.

Case $A(m,7,2)$. In this case $\alpha_2 = \beta_2 = 133.1$ from Table 1. We see that m must be chosen large enough so that $\alpha_1 = \beta_1$ is greater than $360 - (133.1 + 90) = 136.9$. Table 1 shows that this is achieved for $m \geq 8$.

Clearly all groups of the form $A(m,n,2)$ with $m,n \geq 8$ will give 3-complexes with CAT(1) links. Hence the theorem follows. □

We can combine this result with previously known results to get a more complete picture of all 3-generator Artin groups.

Corollary 4.3. *All but finitely many of the 3-generator Artin groups are the fundamental groups of compact locally CAT(0) complexes of dimension at most three.*

Proof. The Artin groups of the form $A(m,n,p)$ where $m,n,p \geq 3$, and the three generator Artin groups with fewer than three relations all admit 2-dimensional CAT(0) structures by the work of [4]. The Artin groups of the form $A(2,2,p)$ $p \geq 2$ all admit 3-dimensional CAT(0) structures by [3]. Theorem 4.2 above covers the remaining cases ($A(m,n,2)$ where $m,n \geq 3$). □

It is worth noting that we could combine the $A(m)$ building blocks to produce compact, locally CAT(0) 3-dimensional Eilenberg Mac Lane complexes for all but finitely many of the 3-generator Artin groups of *large type*; where m, n and p are all at least 3. This is made explicit in Theorem 4.4(1) below. However, in the 3-generator case above, it is more efficient to appeal to the neat 2-dimensional construction of [4].

Note that Theorem 4.2 and Corollary 4.3 exclude precisely 65 infinite type Artin groups $A(m,n,2)$ in addition to the finite types $A(3,3,2)$, $A(4,3,2)$ and $A(5,3,2)$ not considered in the proofs. We note that 3-dimensional locally CAT(0) Eilenberg-Mac Lane complexes have been found for the finite type three generator Artin groups [3]. The CAT(0) dimension is unknown for the remaining 65 examples.

Our 3-dimensional building blocks can also be combined with some limited success in the case of Artin groups with more than 3 generators. Recall that an Artin group is specified by a finite set $\{a_1, \dots, a_n\}$ of generators, and each pair $\{a_i, a_j\}$ of generators satisfies a relation of the form

$$(a_i, a_j)_{m_{ij}} \;=\; (a_j, a_i)_{m_{ij}}$$

where the m_{ij} belong to $\{2,3,4,\dots\} \cup \{\infty\}$. In the case $m_{ij} = \infty$ we have no explicit relation between a_i and a_j. We call the m_{ij} *relator indices.*

Theorem 4.4. *The following types of Artin groups all admit compact, 3-dimensional, locally CAT(0) Eilenberg Mac Lane complexes.*

1. *Artin groups with the property that no triple of relator indices of the form* $\{m_{ij}, m_{jk}, m_{ki}\}$ *appears in the following list*

$\{m,2,2\},\ 2 \leq m < \infty;$	$\{m,3,3\},\ 3 \leq m < 22;$	$\{m,4,4\},\ 4 \leq m < 9;$
$\{m,3,2\},\ 3 \leq m < 44;$	$\{m,4,3\},\ 4 \leq m < 13;$	$\{5,5,4\},\ \{6,5,4\};$
$\{m,4,2\},\ 4 \leq m < 19;$	$\{m,5,3\},\ 5 \leq m < 10;$	
$\{m,5,2\},\ 5 \leq m < 12;$	$\{6,6,3\},\ \{7,6,3\};$	$\{5,5,5\}.$
$\{m,6,2\},\ 6 \leq m < 10;$		
$\{7,7,2\}$		

2. *Artin groups with the property that all the relator indices* m_{ij} *are at least* 4*, and such that if* $\{m_{ij}, m_{jk}, m_{ki}\}$ *is a triple of finite indices, then* $\{m_{ij}, m_{jk}, m_{ki}\} \neq \{4,4,4\}$

Remark. Part 1 of this Theorem includes the class of *triangle free* Artin groups, those with the property that, for each triple of relator indices of the form $\{m_{ij}, m_{jk}, m_{ki}\}$, at least one of the indices is infinite. Triangle free Artin groups were shown to have 2-dimensional locally CAT(0) Eilenberg-MacLane spaces in [4]. However, we note that, for the most part, the Artin groups listed in Theorem 4.4 are not covered by the 2-dimensional results of T. Brady and J. McCammond.

Proof. The proof proceeds in the same way as the proof of Theorem 4.2. We glue the model complexes for the dihedral type Artin groups together, and then check to see if the links in the resulting complex are CAT(1). As before, this reduces to checking that there are no closed loops of length less than 2π in the combined links. Again this reduces to checking that there are no short loops in the corresponding schematic picture of the link.

Since all α and β are at least $\pi/2$, we only have to consider triples of finite indices of the form $\{m_{ij}, m_{jk}, m_{ki}\}$ when looking for short loops in the schematic graph.

In the case of Artin groups of type (1) we use model spaces where $\alpha = \beta$. By employing Table 1 and the approach used in the proof of Theorem 4.2 we find that there is a short loop in the schematic graph associated to a triple $\{m_{ij}, m_{jk}, m_{ki}\}$ of finite indices if and only if it is in the given list.

In the case of Artin groups of type (2) we generally choose $\alpha \neq \beta$. We choose the model space $X_4(\delta)$ for $A(4)$ by letting $\alpha = \alpha_4 = 163°$. Then using Lemma 4.1 (with $m = 4$ we have $\cos\theta = 0$) we have $\beta = \beta_4 = 91.25° > 91°$.

We remark that, regardless of the choice of model space $X_m(\delta)$, $0 < \delta < 90°$, we will always have $\beta > \theta$. (Note that $0 < \theta < 180°$ for all m. From the equation $\cos\beta = \sin^2\delta\cos\theta - \cos^2\delta$ in the proof of Lemma 4.1, we have that $\cos\beta = \cos\theta$ if and only if $\delta = \pm 90°$, while at $\delta = 0°$, $\cos\beta = -1$. So, by continuity, $\cos\beta < \cos\theta$ for δ in the given range, and hence $\beta > \theta$.) Thus, when $m \geq 5$ we will always have $\beta > 108°$, leaving us free to choose α as close to $180°$ as we like. We choose the spaces $X_m(\delta)$ for $m \geq 5$ by putting $\alpha = \alpha_{\geq 5} = 179°$ ($\delta = 89.5°$), so ensuring that $\alpha_{\geq 5} + 2\beta_4 > 360°$.

Again we consider the triples $\{m_{ij}, m_{jk}, m_{ki}\}$ of finite indices, all at least 4. Clearly the sum of three α edges will always be greater than $3(163°) > 360°$. Otherwise, we must consider the sum of one α edge with two β edges (one edge coming from the schematic link component associated with each of the three relator indices in $\{m_{ij}, m_{jk}, m_{ki}\}$). If one of the β edges comes from an index greater than 5 (i.e: its length is greater than 108°), then the sum of the three will be greater than $163° + 91° + 108° = 362°$. This leaves the case where both β edges are short (coming from relator indices equal to 4). We have already seen that $\alpha_{\geq 5} + 2\beta_4 > 360°$, leaving only the possibility that $\{m_{ij}, m_{jk}, m_{ki}\} = \{4, 4, 4\}$ which is excluded by the hypothesis. We note, unfortunately, that there is no possible choice of α_4 which allows $\alpha_4 + 2\beta_4$ to exceed 360°. □

5 Related results and further questions

Perhaps the most obvious:

Question 1. Do any of the groups $A(m, 2n, 2)$, with $m, n \geq 2$, have CAT(0) dimension 2? Optimistically, the answer might be "no".

We remarked in Section 4 after the proof of Theorem 4.2 and Corollary 4.3 that our 3-dimensional constructions did not give CAT(0) structures for 65 Artin groups of type $A(m, n, 2)$ which are not finite type. These are $A(3, n, 2)$ for $6 \leq n \leq 43$, $A(m, 4, 2)$ for $4 \leq m \leq 18$, $A(m, 5, 2)$ for $5 \leq m \leq 11$, $A(m, 6, 2)$ for $6 \leq m \leq 9$, and $A(7, 7, 2)$.

Question 2. What is the CAT(0) dimension of each of the 65 Artin groups listed above?

There is some promise that our techniques may be used in higher dimensions, either to demonstrate dimension gaps of more than one for these (or other) 2-dimensional groups, or to demostrate dimension gaps for groups (perhaps Artin groups) of geometric dimension greater than 2. Comparing the 2-dimensional constructions of [4] with our Theorem 4.4 suggests many examples of Artin groups with 4 or more generators which may be worth investigating.

The motivation for the 2-dimensional structures found in [4] was to show that the associated Artin groups are biautomatic.

Question 3. Can one use the 3-dimensional CAT(0) structures to show that the Artin groups listed in Theorem 4.4 are biautomatic? Those of Theorem 4.4 (2) are contained in the class of extra-large type Artin groups (all $m_{i,j} \geq 4$) which have already been shown to be biautomatic by D. Peifer [13]. According to Corollary III.Γ.4.8 of [7], any group which acts properly and cocompactly on a CAT(0) space is semi-hyperbolic (asynchronously bi-combable) and hence has solvable word and conjugacy problems.

There are other variations of the notion of CAT(0) dimension and many interesting questions to consider. For example, one could define the CAT(-1) dimension of a group exactly as we defined CAT(0) dimension in this paper by replacing the occurrences of CAT(0) with CAT(-1).

Question 4. Are there torsion free Gromov hyperbolic groups with distinct CAT(0) and CAT(-1) dimensions? The answer is yes for hyperbolic groups with torsion (see M. Kapovich (preprint, Utah), also [2]).

It is unknown if Gromov hyperbolic groups are all CAT(0) or CAT(-1), that is whether they act properly and cocompactly on a finite dimensional CAT(0) complex. Perhaps the following question is more manageable. Note that we do not require cocompactness in the definitions of CAT(0) and CAT(-1) dimensions.

Question 5. Do all torsion free Gromov hyperbolic groups have finite CAT(0) or CAT(-1) dimensions?

We know that the Rips complex gives a finite Eilenberg-MacLane space for torsion free Gromov hyperbolic groups, so they have finite geometric dimension.

Of course our definition of CAT(0) dimension does not restrict us to looking at torsion free groups. Question 5, for example, is still of considerable interest in the case of hyperbolic groups with torsion. Some interesting examples of hyperbolic groups with torsion are considered in [2]. Here the CAT(0) dimension is compared with the minimal dimension of a proper G-space, an approach which is also in the spirit of finding the minimum dimension of a nonpositively curved $\underline{\mathrm{E}}(G)$.

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