Solutions to 5.2

5.2. Let \( c \) be a constant and consider the density function
\[
f(y) = \begin{cases} 
  c(2-y) & \text{if } 0 \leq y \leq 1 \\
  0 & \text{elsewhere}
\end{cases}
\]

a. Find the value of \( c \).
Solution:
We know that \( \int_{-\infty}^{\infty} f(y) \, dy = 1 \) since this is a property of legitimate probability density functions.
This gives us
\[
1 = \int_{-\infty}^{\infty} f(y) \, dy = \int_{0}^{1} c(2-y) \, dy = \left[ 2cy - \frac{cy^2}{2} \right]_{y=0}^{y=1} = 2c - \frac{c}{2} = \frac{3c}{2}
\]
\[
\frac{3c}{2} = 1 \\
c = \frac{2}{3}
\]
b. Find the cumulative distribution function \( F(y) \).
Solution:
We know that \( F(y) = \int_{-\infty}^{y} f(t) \, dt \) where \( t \) is a dummy variable that disappears with integration; any other variable name could be used instead. This gives
\[
F(y) = \int_{-\infty}^{y} f(t) \, dt = \int_{0}^{2} \frac{2}{3}(2-t) \, dt = \left[ 4t - \frac{t^2}{3} \right]_{t=0}^{t=y} = \frac{4y}{3} - \frac{y^2}{3} = \frac{1}{3} (4y - y^2)
\]
for \( 0 \leq y \leq 1 \). So we have
\[
F(y) = \begin{cases} 
  \frac{1}{3} (4y - y^2) & \text{if } 0 \leq y \leq 1 \\
  0 & \text{elsewhere}
\end{cases}
\]
c. Compute \( F(.4) \).
Solution:
We know from (b) that
\[
F(y) = \begin{cases} 
  \frac{1}{3} (4y - y^2) & \text{if } 0 \leq y \leq 1 \\
  0 & \text{elsewhere}
\end{cases}
\]
so
\[
F(.4) = \frac{1}{3} (4(.4) - (.4)^2) = \frac{1}{3} (1.6 - .16) = \frac{1}{3} (1.44) = .48
\]
since \( 0 \leq .4 \leq 1 \).
d. Compute \( P(.1 \leq y \leq .6) \).
Solution:
Notice that \( F(.1) = P(y \leq .1) \) and \( F(.6) = P(y \leq .6) = P(y \leq .1) + P(.1 \leq y \leq .6) \) so
\[
F(.6) = P(y \leq .1) + P(.1 \leq y \leq .6)
\]
\[
F(.6) = F(.1) + P(.1 \leq y \leq .6)
\]
\[
P(.1 \leq y \leq .6) = F(.6) - F(.1)
\]
We also know that
\[ F(0.1) = \frac{1}{3} (4(0.1) - (0.1)^2) = \frac{1}{3} (0.4 - 0.01) = \frac{1}{3} (0.39) = 0.13 \text{ since } 0 \leq 0.1 \leq 1, \text{ and} \]
\[ F(0.6) = \frac{1}{3} (4(0.6) - (0.6)^2) = \frac{1}{3} (2.4 - 0.36) = \frac{1}{3} (2.04) = 0.68 \text{ since } 0 \leq 0.6 \leq 1, \]
so
\[ P(0.1 \leq y \leq 0.6) = F(0.6) - F(0.1) = 0.68 - 0.13 = 0.55 \]