Group Theory Problems

MATH 5363, Spring 2005

1. Prove that if $G/Z(G)$ is cyclic, then G is abelian.
2. If $|G| = p^n$, p-prime, then $Z(G) \neq < e >$.
3. If $|G| = pn$ with $p > n$, p prime and H is a subgroup of order p, then H is normal in G.
4. Let G be a group, $|G| = 36$. Prove that G has a non-trivial normal subgroup.
5. Let G be a group of order 30. Show that either a 3-Sylow or a 5-Sylow subgroup must be normal in G.
6. Let $|G| = pq$, p and q distinct primes, $p < q$. Show that if $p \not| (q-1)$, then G is cyclic.
7. Prove that the group of permutations of 3 elements S_3 is not the direct product of any family of its proper subgroups.
8. Let G, H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $(|G|, |H|) = 1$.
9. If N is a normal subgroup of G and $N \cap [G, G] =< e >$, then $N \subset Z(G)$.
10. Suppose G is a finite group and all its Sylow subgroups are normal. Show that G is a direct product of its Sylow subgroups.