Test II

MATH 5353

November 29, 2004

1. If H is a finite index subgroup in a group G, prove that there is a subgroup N of G contained in H and of finite index in G such that $aNa^{-1} = N$ for all $a \in G$.

2. If N is a normal subgroup of a group G, N is finite, H is a subgroup of G of finite index and the index $[G : H]$ and the order $|N|$ are relatively prime, prove that $N \subset H$.

3. Let $\alpha = (12)(34)$ and $\beta = (24)$. Show that the group generated by α and β is isomorphic to D_4.

4. For each bilinear symmetric function $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ below given by its matrix in the standard basis determine the dimension of the kernel and the maximal dimension of a subspace on which the function is positive definite.

   a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

   b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

   c) $\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$

5. Consider $V = \mathbb{C}$ as a vector space of dimension 2 over $\mathbb{R}$.

   a) Show that $\alpha : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by $\alpha(z, w) = Re(z\bar{w})$ is a positive definite bilinear symmetric function.

   b) Let $z \in \mathbb{C}$ and $L_z : \mathbb{C} \to \mathbb{C}$ be the map $w \to zw$. What is the matrix of L_z with respect to the basis $(1, i)$ of $\mathbb{C}$ over $\mathbb{R}$?

   c) For which complex numbers z do we have $\alpha(L_z(w_1), L_z(w_2)) = \alpha(w_1, w_2)$ for any $w_1, w_2 \in \mathbb{C}$?