NAME: Solutions

Math 3333, Test II

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Show all your work to receive full credit. The use of books and notes is not allowed. Good luck!

I. (20 pts) Which of the following subsets of the vector space $\mathbb{R}^4$ are subspaces? Explain your answers.

a) The set of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, such that $a = 1$, $b = 0$.

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \\ c \\ d \end{bmatrix} \middle| c, d \in \mathbb{R} \right\} \qquad \begin{bmatrix} 1 \\ 0 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ c_1+c_2 \\ d_1+d_2 \end{bmatrix}$$

not closed under +

not a subspace

b) The set of vectors $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, such that $c = a + 2b$, $d = 0$.

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+2b \\ 0 \end{bmatrix} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

We showed that span of several vectors is a subspace.

II. (20 pts) Determine whether the vector $p(t) = t^2 - 3t + 2$ in P_2 belongs to the span of $p_1(t) = t^2 + t$, $p_2(t) = t + 1$ and $p_3(t) = t^2 + 1$

Are there numbers a, b, c such that

$a p_1(t) + b p_2(t) + c p_3(t) = p(t)$?

$a t^2 + at + bt + b + ct^2 + c =$

$= (a+c)t^2 + (a+b)t + b + c = t^2 - 3t + 2$

$a + c = 1$

$a + b = -3$

$b + c = 2$

We wish to solve this system for a, b, c.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & 1 & 2 \end{array}\right] \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\leftarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$a + c = 1$
$b - c = -4$
$c = 3$

Then $b = -1$
$a = -2$

So $p(t) = -2p_1(t) - p_2(t) + 3p_3(t)$

and is in the span of p_1, p_2, p_3.

III. (20 pts) Find a basis for $\mathbb{R}^3$ that includes the vector $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Consider the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

span $S = \mathbb{R}^3$, since S contains the standard basis.

S is linearly dependent.

Then one of the vectors in S is a linear combination of the previous ones.

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Consider the set T obtained from S by deleting $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, i.e $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

span $T = \mathbb{R}^3$

T is linearly independent set, since

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} = 2 \neq 0.$$

Thus T is a basis of $\mathbb{R}^3$ that contains $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

IV. (20 pts) Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 + 4x_5 &= 0 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 0 \\ x_1 - x_2 + 3x_3 + 5x_4 + 6x_5 &= 0 \end{aligned}$$

Bring the coefficient matrix to the reduced row echelon form

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ -1 & 2 & 3 & 4 & 5 \\ 1 & -1 & 3 & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 7 & 9 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -4 & -1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}$$

$x_1 - 4x_4 - x_5 = 0$ or $x_1 = 4x_4 + x_5$

$x_2 - 3x_4 - x_5 = 0$ $\quad x_2 = 3x_4 + x_5$

$x_3 + 2x_4 + 2x_5 = 0$ $\quad x_3 = -2x_4 - 2x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4s+t \\ 3s+t \\ -2s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 4 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of the solution space.

Thus its dimension is 2.

V. (20 pts)

Let $S = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$ and $T = \{\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\}$ be ordered bases of $\mathbb{R}^2$. If v is in $\mathbb{R}^2$ and $[v]_T = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, determine $[v]_S$.

Find the transition matrix $P_{S \leftarrow T}$

$$\begin{bmatrix} 1 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$c_1 + c_2 = 1$ $\qquad -3c_2 = -6, \; c_2 = 2$

$c_1 - 2c_2 = -5$ $\qquad$ then $c_1 = -1$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$P_{S \leftarrow T} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$[v]_S = P_{S \leftarrow T} [v]_T$$

$$[v]_S = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$