NAME: Solutions

Math 3113, Test II

October 20, 2009

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I. (20 pts) Find the general solution of a homogeneous equation

$y^{(4)} = 16y$

the characteristic eqn. is

$r^4 - 16 = 0.$

$r^4 - 16 = (r^2-4)(r^2+4) = (r-2)(r+2)(r-2i)(r+2i)$

$r_1 = 2,\ r_2 = -2,\ r_3 = 2i,\ r_4 = -2i$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x}$$

II. (20 pts) Solve the initial value problem

$$y'' - 4y' + 3y = 0; y(0) = 7; y'(0) = 11$$

the char. eqn. is

$$r^2 - 4r + 3 = 0$$

$$r_1 = 1, \quad r_2 = 3$$

the general solution is

$$y = c_1 e^x + c_2 e^{3x}$$

$$y' = c_1 e^x + 3c_2 e^{3x}$$

$$y(0) = c_1 + c_2 = 7$$

$$y'(0) = c_1 + 3c_2 = 11$$

subtract 1st eqn. from 2nd to get

$$2c_2 = 4 \quad \text{or} \quad c_2 = 2$$

Substitute into 1st eqn. to get

$$c_1 = 5$$

So the particular solution satisfying given initial conditions is

$$\boxed{y = 5e^x + 2e^{3x}}$$

III. (20 pts) Find the general solution of $y^{(3)} + 3y'' - 54y = 0$, if you know that one solution is $y = e^{3x}$.

the characteristic eqn. is

$$r^3 + 3r^2 - 54 = 0$$

$y = e^{3x}$ is a solution implies that $r_1 = 3$ is one root of the char. eqn.

We need to find the other two roots.

$$
\begin{array}{r|l}
 & r^2 + 6r + 18 \\
r-3 & r^3 + 3r^2 - 54 \\
 & -(r^3 - 3r^2) \\
\hline
 & 6r^2 - 54 \\
 & -(6r^2 - 18r) \\
\hline
 & 18r - 54 \\
 & -(18r - 54) \\
\hline
 & 0
\end{array}
$$

So

$$r^3 + 3r^2 - 54 = (r-3)(r^2 + 6r + 18)$$

the roots of $r^2 + 6r + 18$ are

$$r_{2,3} = \frac{-6 \pm \sqrt{36 - 72}}{2} = \frac{-6 \pm 6i}{2} = -3 \pm 3i$$

the general solution is

$$\boxed{y = c_1 e^{3x} + e^{-3x}(c_2 \cos 3x + c_3 \sin 3x)}$$

IV. (20 pts) Find a particular solution y_P of

$y^{(3)} - y = e^x + 7$

$f(x) = e^x + 7$, e^x, 1

First ~~approx~~. guess at a particular solution is

$$y_p = A + Be^x$$

Need to check if either 1 or e^x are solutions of

$$y^{(3)} - y = 0.$$

$$r^3 - 1 = 0.$$

$$r_1 = 1, \quad r_{2,3} = \frac{1 \pm \sqrt{3}\,i}{2}$$

So $y = e^x$ is a solution of $y^{(3)} - y = 0$.

Adjust our initial guess

$$y_p = A + Bxe^x$$

$$y_p' = B(e^x + xe^x), \quad y_p'' = B(2e^x + xe^x), \quad y_p^{(3)} = B(3e^x + xe^x)$$

$$y_p^{(3)} - y = B(3e^x + xe^x) - A - Bxe^x = -A + 3Be^x = e^x + 7$$

So $A = -7$, $3B = 1$ or $B = \frac{1}{3}$

$$\boxed{y_p = -7 + \tfrac{1}{3}xe^x}$$

V. (20 pts) Solve the initial value problem

$$y'' + 3y' + 2y = 6e^x; y(0) = 0; y'(0) = 3$$

First find a particular solution of $y'' + 3y' + 2y = 6e^x$

$f(x) = 6e^x$

We guess $y_p = Ae^x$

Check that this y_p does not satisfy $y'' + 3y' + 2y = 0$

$r^2 + 3r + 2 = 0$

$r_1 = -1, \ r_2 = -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

Find A: $y_p = Ae^x = y_p' = y_p''$

$y_p'' + 3y_p' + 2y = 6Ae^x = 6e^x$

so $A = 1$

$y_p = e^x$

However this particular solution does not satisfy the given initial conditions. So we need to find another particular sol.

$y = e^x + c_1 e^{-x} + c_2 e^{-2x}$, $y(0) = 1 + c_1 + c_2 = 0$

$y' = e^x - c_1 e^{-x} - 2c_2 e^{-2x}$, $y'(0) = 1 - c_1 - 2c_2 = 3$

add both eqns to get

$2 - c_2 = 3$, $c_2 = -1$

$c_1 = -1 - c_2 = 0.$

$$\boxed{y = e^x - e^{-2x}}$$