NAME: Solutions

Math 3113, Test I

September 22, 2009

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I. (20 pts) Find the position function $x(t)$ of a moving particle with acceleration $a(t) = \frac{1}{\sqrt{t+4}}$, initial position $x_0 = x(0) = 1$ and initial velocity $v_0 = v(0) = -1$.

$$a(t) = \frac{1}{\sqrt{t+4}} = (t+4)^{-1/2}$$

$$v(t) = \int (t+4)^{-1/2}\,dt = 2(t+4)^{1/2} + C_1$$

$$v(0) = -1 \Rightarrow C_1 = -5$$

$$v(t) = 2(t+4)^{1/2} - 5$$

$$x(t) = \int 2(t+4)^{1/2} - 5\,dt = \frac{4}{3}(t+4)^{3/2} - 5t + C_2$$

$$x(0) = \frac{32}{3} + C_2 = 1, \quad C_2 = -\frac{29}{3}$$

$$\boxed{x(t) = \frac{4}{3}(t+4)^{3/2} - 5t - \frac{29}{3}}$$

II. (20 pts) Bacteria grow in a certain culture at a rate proportional to the amount present. Suppose there are 100 bacteria present initially and the amount doubles in an hour. How many will there be in additional half an hour?

$P(t) = P_0 e^{kt}$

$P_0 = 100$

$P(1) = 2P_0$ i.e $2\cancel{P_0} = \cancel{P_0} e^{k}$

$k = \ln 2$

$P(t) = 100 e^{\ln 2 \cdot t}$

$P(\frac{3}{2}) = 100 e^{\ln 2 \cdot \frac{3}{2}} \approx 283$ bacteria

III. (20 pts) Sugar dissolves in water at a rate proportional to the amount still undissolved. Suppose there are 50 grams of sugar initially and after 3 minutes only 20 grams are left. What is the time it takes for 90% of the sugar to dissolve?

$$N(t) = N_0 e^{kt}$$

$$N_0 = 50$$

$$20 = N(3) = 50 e^{3k}$$

$$k = \frac{1}{3} \ln 0.4 \approx -0.305$$

When 90% of sugar dissolve, 10% of 50 grams is left. So

$$5 = 50 e^{\frac{1}{3} \ln 0.4 \cdot t}$$

$$\ln 0.1 = \frac{1}{3} \ln 0.4 \cdot t$$

$$t = \frac{3 \ln 0.1}{\ln 0.4} \approx 7.5 \text{ minutes}$$

IV. (20 pts) Find a general solution of a differential equation.

a) $xy\frac{dy}{dx} = y^2 + x\sqrt{4x^2+y^2}$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{4x^2+y^2}\;{}^{1/x}}{y\;{}_{1/x}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{4+(y/x)^2}}{y/x}$$

$$v = \frac{y}{x},\ y = vx,\ \frac{dy}{dx} = \frac{dv}{dx}\cdot x + v$$

$$\frac{dv}{dx}\cdot x + \cancel{v} = \cancel{v} + \frac{\sqrt{4+v^2}}{v}$$

separable eqn.

$$\frac{v\,dv}{\sqrt{4+v^2}} = \frac{dx}{x}$$

integrate to get

$$(4+v^2)^{1/2} = \ln|x| + C$$

$$\sqrt{4+\left(\frac{y}{x}\right)^2} = \ln|x| + C$$

$$\boxed{\sqrt{4x^2+y^2} = |x|\ln|x| + C|x|}$$

b) $3y^2\frac{dy}{dx} + y^3 = e^{-x}$

$$\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}e^{-x}y^{-2}$$

Bernoulli, $n=-2$

$$v = y^3,\quad y = v^{1/3},\quad \frac{dy}{dx} = \frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx}$$

$$\frac{1}{3}v^{-\frac{2}{3}}\frac{dv}{dx} + \frac{1}{3}v^{1/3} = \frac{1}{3}e^{-x}y^{-2}$$

multiply by $3v^{2/3}$ to get

$$\frac{dv}{dx} + v = e^{-x}$$

linear 1st order in v

$$\rho(x) = e^{\int 1\,dx} = e^x$$

$$ve^x = \int e^{-x}\cdot e^x\,dx = x + C$$

$$v = xe^{-x} + Ce^{-x}$$

$$y^3 = xe^{-x} + Ce^{-x}$$

$$\boxed{y = \sqrt[3]{xe^{-x} + Ce^{-x}}}$$

V. (20 pts) Solve the initial value problem.

a) $x\frac{dy}{dx} - 3y = x^3$, $y(1) = 10$

$\frac{dy}{dx} - \frac{3}{x}y = x^2$

linear 1st. order

$p(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$

$\boxed{y = x^3 \ln x + 10x^3}$

$y x^{-3} = \int x^2 \cdot x^{-3}\, dx = \ln x + C$

$y = x^3 \ln x + Cx^3$

$y(1) = C = 10$

b) $\frac{dy}{dx} = 6e^{2x-y}$, $y(0) = 0$

$\frac{dy}{dx} = 6e^{2x} \cdot e^{-y}$

$e^y\, dy = 6e^{2x}\, dx$

$\boxed{y = \ln(3e^{2x} - 2)}$

$e^y = 3e^{2x} + C$

$y = \ln(3e^{2x} + C)$

$y(0) = \ln(3 + C) = 0$

$3 + C = 1$

$C = -2$