Project I: Moving a planar robot arm

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1 The Problem

Many industrial processes are carried out by computer-controlled robots. The design and control of robots is the subject of a discipline called robotics, which makes heavy use of mathematics, including calculus. We will discuss the control of motion of a very simple 2-dimensional, 2-joint robot arm. There are few, if any, mechanisms of this type. However, the concepts we will develop in this simple example are quite similar to those one must wrestle with for more realistic robots.

The planar robot has two links, each of which is a line segment. Point B, the base of the robot is fixed. The first link rotates around point B, and the second link rotates with respect to the first around point C. The entire motion of the robot takes place in the plane. In this plane we will choose a coordinate system whose origin is at B, and whose positive x-axis points to the right. The angle θ_1 between the positive x-axis and link 1, and the angle θ_2 between links 1 and 2, are both controlled by the robot's computer.

The part of the robot which does useful work is at the tip, called the *end effector*. One might imagine a drill or gripper or paint sprayer attached there. Usually, a particular point D on the end effector is singled out for attention. The key problem in robotics is "How to move the end effector in its work area?" We cannot control the end effector directly; instead we must control the angles θ_1 and θ_2 so as to create the desired motion at the end effector. This is a problem for calculus.

2 The Kinematic Equations

We will start with a static problem: given particular values for θ_1 and θ_2, what are the resulting values for x_D and y_D, the coordinates of the end effector point D? This is called the *forward kinematic problem.*

Let us suppose that the lengths of links 1 and 2 are l_1 and l_2 respectively. We first find the coordinates x_C and y_C of point C:

$$x_C = l_1 \cos\theta_1, y_C = l_1 \sin\theta_1 \tag{1}$$

The coordinates of the point D are then obtained:

$$\begin{aligned} x_D &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y_D &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned} \tag{2}$$

Robots usually have measuring devices to measure the angles at the joints. From these angles, and knowing l_1 and l_2, the robots computer can use equations (2) to calculate the position of the end effector at any given instant.

Often we want to turn the forward kinematic process around: given a point (x, y) in the robot's work area, what values of θ_1 and θ_2 are required to put the end effector at (x, y)? The answer is provided by *inverse kinematic equations*. There are normally two possible solutions, one called *elbow regular* with $\theta_2 > 0$, and the other called *elbow irregular* with $\theta_2 < 0$.

EXERCISES:

1. Describe the work area, i.e. the region of the plane which can be reached by the end effector a) when $l_1 > l_2$, b) when $l_1 < l_2$.

2. Derive the inverse kinematic equations. Here is a suggested path.

- In equations (2) solve for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$, substitute solutions into $\sin^2(\theta_1 + \theta_2) + \cos^2(\theta_1 + \theta_2) = 1$, and simplify to get

  $$y \sin\theta_1 + x \cos\theta_1 = A,$$

  where $A = \frac{x^2+y^2+l_1^2-l_2^2}{2l_1}$.

- Solve the above equation along with $\sin^2\theta_1 + \cos^2\theta_1 = 1$, getting

  $$\cos\theta_1 = \frac{Ax \pm y\sqrt{x^2 + y^2 - A^2}}{x^2 + y^2}$$

Note that because $\cos\theta_1 = \cos(-\theta_1)$ this usually gives four possible values for θ_1. One corresponds to the elbow regular position, one to the elbow irregular position, and two are extraneous roots. Once θ_1 is determined, it is easy to find θ_2 using equations (2).

3 Velocity Control

Suppose we have a point $T = (x, y)$ in mind and we wish the end effector to pass through it with certain horizontal and vertical velocities. The way to think about it mathematically is to imagine the end effector moving along a curve parameterized by time t. In other words, the curve γ is specified by two functions $x(t)$ and $y(t)$, which give the coordinates of the end effector at time t. A familiar result from calculus is that the velocity vector of the end effector at time t is the tangent vector $\gamma' = (x'(t), y'(t))$.

For example, suppose we are washing a vertical window. We would like there to be no motion in the x-direction, i.e. $x'(t) = 0$. We want to be sure the vertical velocity is not too fast, so the window really gets clean, and not too slow so the job doesn't take longer than it has to. For example, suppose we want to be moving upward at one foot per second, i.e. $y'(t) = 1$.

With x, y, θ_1, θ_2 written as functions of time, we can differentiate the equations (2) with respect to t to get

$$\begin{aligned} x' &= -l_1 \sin(\theta_1)\theta_1' - l_2 \sin(\theta_1 + \theta_2)(\theta_1' + \theta_2') \\ y' &= l_1 \cos(\theta_1)\theta_1' + l_2 \cos(\theta_1 + \theta_2)(\theta_1' + \theta_2') \end{aligned}$$

Doing a little algebra and using equations (2), this can be simplified to

$$\begin{aligned} x' &= -y\theta_1' - l_2 \sin(\theta_1 + \theta_2)\theta_2' \\ y' &= x\theta_1' + l_2 \cos(\theta_1 + \theta_2)\theta_2' \end{aligned} \tag{3}$$

In our window-washing example, x' and y' have been chosen to be 0 and 1. The point (x, y) is known, so we can apply the inverse kinematic equations to find θ_1 and θ_2. Hence he only unknowns in (3) are θ_1' and θ_2', and we will be able to solve for them.

For example, if $l_1 = 3.0$, $l_2 = 2.0$, $x = 4.0531$, $y = 1.6037$ then the inverse kinematic equations would give (Exercise 3)

$$\theta_1 = 0.7854rad(45^\circ), \theta_2 = -1.0472rad(-60^\circ)$$

This is elbow irregular configuration. Then we obtain the following system of equations for the angle velocities:

$$\begin{aligned} 0 &= -1.6037\theta_1' + 0.5176\theta_2' \\ 1 &= 4.0531\theta_1' + 1.9318\theta_2' \end{aligned}$$

Solving this system gives $\theta_1' = 0.0996rad/sec$, $\theta_2' = 0.3086rad/sec$.

EXERCISES:

3. Use the result of Exercise 2 to verify the values of θ_1 and θ_2 for the elbow irregular position in the vertical window-washing example.

4. Use the result of Exercise 2 to find the values of θ_1 and θ_2 for the elbow regular position for the same example. (Drawing a careful picture will help pick out the correct value for θ_1.) Solve for the values of θ_1' and θ_2' needed to get the motion $x' = 0$, $y' = 1$ in the elbow regular position.