Math 2423, Test II

November 11, 2005

Show all your work to receive full credit. You can use a 3 x 5 index card with favorite formulas. The use of books, lecture notes and calculators is not allowed. Good luck!

I. (12 pts) Find the formula for the inverse $g = f^{-1}$ of the function $f(x) = \frac{4x-1}{2x+3}$. State the domain and the range of g.

$$y = \frac{4x-1}{2x+3}$$

$$y(2x+3) = 4x-1.$$

$$2yx + 3y = 4x - 1$$

$$(2y-4)x = -3y-1$$

$$x = -\frac{1+3y}{2y-4.}$$

$$\boxed{y = -\frac{1+3x}{2x-4}}$$

$$D(f^{-1}) = (-\infty, 2) \cup (2, +\infty).$$

$$R(f^{-1}) = (-\infty, -\tfrac{3}{2}) \cup (-\tfrac{3}{2}, +\infty).$$

II. (20 pts) Find $\frac{dy}{dx}$

a) $y = e^{\tan x}$;

$$\frac{dy}{dx} = e^{\tan x} \cdot (\tan x)' = e^{\tan x} \cdot \sec^2 x$$

b) $y = \ln(xe^x)$;

$$\frac{dy}{dx} = \frac{1}{xe^x} \cdot (xe^x)' = \frac{e^x + xe^x}{xe^x} = \frac{1}{x} + 1$$

c) $y = ln(\arctan x) + e^{\arcsin x}$;

$$\frac{dy}{dx} = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2} + e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}$$

d) $y = x^{\ln x}$.

$$y = (e^{\ln x})^{\ln x} = e^{(\ln x)^2}$$

$$\frac{dy}{dx} = e^{(\ln x)^2} \cdot 2\ln x \cdot \frac{1}{x}$$

$$= x^{\ln x} \cdot \frac{2\ln x}{x}.$$

III. (20 pts)

a) Find the exact value of $e^{3\ln 2}$.

$$e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$$

b) Solve the equation for x

$$\ln(2x+1) = 2 - \ln x$$

$$\ln(2x+1) + \ln x = 2$$

$$\ln(x \cdot (2x+1)) = 2$$

$$2x^2 + x = e^2$$

$$2x^2 + x - e^2 = 0$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+8e^2}}{4}.$$

we need $x > 0$ so

$$x = \frac{-1 + \sqrt{1+8e^2}}{4}$$

IV. (20 pts) Find the limit

a) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \underset{\text{L'Hosp.}}{=} \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} =$$

$\frac{\infty}{\infty}$ $\qquad$ $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{2\ln x}{x} \underset{\text{L'Hosp.}}{=} \lim_{x\to\infty} \frac{2}{x} = 0.$$

$\frac{\infty}{\infty}$

b) $\lim_{x\to\infty} x\tan(1/x)$

$$\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \frac{\sin 1/x}{1/x \cdot \cos 1/x} =$$

$$= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{y\to 0} \frac{\sin y}{y} \underset{\text{L'Hosp.}}{=}$$

$y = \frac{1}{x}$ $\qquad$ $\frac{0}{0}$

$$= \lim_{y\to 0} \frac{\cos y}{1} = 1.$$

V. (28 pts) Integrate

a) $\int x^2 \cos x dx$

$u = x^2 \qquad dv = \cos x\, dx$

$du = 2x\, dx \qquad v = \sin x$

$= x^2 \sin x - 2\int x \sin x\, dx$

$u = x \qquad dv = \sin x\, dx$

$du = dx \qquad v = -\cos x$

$= x^2 \sin x - 2\left[-x\cos x + \int \cos x\, dx\right]$

$= x^2 \sin x + 2x\cos x - 2\int \cos x\, dx$

$= x^2 \sin x + 2x\cos x - 2\sin x + C.$

b) $\int \sec^4 x dx$ $= \int \sec^2 x \cdot \sec^2 x\, dx =$

$= \int (\tan^2 x + 1)\sec^2 x\, dx = \int (u^2 + 1)\, du$

$u = \tan x$

$du = \sec^2 x\, dx$

$= \frac{u^3}{3} + u + C = \frac{\tan^3 x}{3} + \tan x + C.$

c) $\int \frac{\sqrt{x^2-1}}{x}dx = \int \frac{\sqrt{\sec^2 t - 1}}{\sec t} \tan t \cdot \sec t \, dt =$

$x = \sec t$

$dx = \tan t \cdot \sec t \, dt$

$$= \int \tan^2 t \, dt = \int (\sec^2 t - 1) \, dt =$$

$$= \int \sec^2 t \, dt - \int dt = \tan t - t + C$$

d) $\int \frac{8}{(x-2)(x^2+4)}dx$

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+(Bx+C)(x-2)}{(x-2)(x^2+4)} =$$

$$= \frac{Ax^2+4A + Bx^2 - 2Bx + Cx - 2C}{(x-2)(x^2+4)} =$$

$$= \frac{(A+B)x^2 + (-2B+C)x + (4A-2C)}{(x-2)(x^2+4)} = \frac{8}{(x-2)(x^2+4)}$$

$$\begin{cases} A+B=0 \\ -2B+C=0. \\ 2A-C=4 \end{cases} \qquad \begin{matrix} A=-B \\ C=2B. \\ -4B=4 \end{matrix} \qquad \begin{matrix} B=-1 \\ A=1 \\ C=-2 \end{matrix}$$

$$\int \frac{8}{(x-2)(x^2+4)}dx = \int \frac{1}{x-2} - \frac{x+2}{x^2+4} \, dx =$$

$$= \int \frac{1}{x-2}dx - \int \frac{x}{x^2+4}dx - 2\int \frac{dx}{x^2+4} =$$

$$= \ln|x-2| - \int \frac{x\,dx}{x^2+4} - \int \frac{dx}{\left(\frac{x}{2}\right)^2+1}$$

$u = x^2+4$

$du = 2x\,dx.$

$v = \frac{x}{2}$

$dv = \frac{1}{2}\,dx$

$$= \ln|x-2| - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{dv}{v^2+1}$$

$$= \ln|x-2| - \frac{1}{2}\ln|u| - \frac{1}{2}\arctan v + C$$

$$= \ln|x-2| - \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C.$$