Contemporary Mathematics

On Kostant's Theorem for Lie Algebra Cohomology

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1. Introduction

1.1. In 1961, Kostant proved a celebrated result which computes the ordinary Lie algebra cohomology for the nilradical of the Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$ with coefficients in a finite-dimensional simple $\mathfrak{g}$-module. Over the last forty years other proofs have been discovered. One such proof uses the properties of the Casimir operator on cohomology described by the Casselman-Osborne theorem (cf. [GW, §7.3] for details). Another proof uses the construction of BGG resolutions for simple finite-dimensional $\mathfrak{g}$-modules [Ro]. Recently, Polo and Tilouine [PT] constructed BGG resolutions over $\mathbb{Z}_{(p)}$ for finite-dimensional irreducible G-modules where G is a semisimple algebraic group with high weights in the bottom alcove as long as $p \geq h-1$ (h is the Coxeter number for the underlying root system). One can then use a base change argument to show that Kostant's theorem holds for these modules over algebraically closed fields of characteristic p when $p \geq h-1$. It should be noted that Friedlander and Parshall had earlier obtained a slightly weaker formulation of this result (cf. [FP1, §2])

The aim of this paper is to investigate and compare the cohomology of the unipotent radical of parabolic subalgebras over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$. We present a new proof of Kostant's theorem and Polo-Tilouine's extension in Sections 2–4. Our proof employs known linkage results in Category $\mathcal{O}_J$ and the graded G_1T category for the first Frobenius kernel G_1. There are several advantages to our approach. Our proofs of these cohomology formulas are self-contained and our approach is presented in a conceptual manner. This enables us to identify key issues in attempting to compute these cohomology groups for small primes.

In Section 5, we prove that when $p < h-1$, there are always additional cohomology classes in $\mathrm{H}^\bullet(\mathfrak{u}, \overline{\mathbb{F}}_p)$ beyond those given by Kostant's formula. The proof of this result relies heavily on the modular representation theory of reductive algebraic groups. Furthermore, we exhibit natural classes that arise in $\mathrm{H}^{2p-1}(\mathfrak{u}, \overline{\mathbb{F}}_p)$ when $\Phi = A_{p+1}$ which do not arise over fields of characteristic zero. In Section 6, we examine several low rank examples of $\mathrm{H}^\bullet(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$

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which were generated using MAGMA. These examples suggest interesting phenomena which lead us to pose several open questions in Section 7.

1.2. Notation. The notation and conventions of this paper will follow those given in [Jan]. Let k be an algebraically closed field, and G a simple algebraic group defined over k with T a maximal torus of G. The root system associated to the pair (G, T) is denoted by Φ. Let Φ^+ be a set of positive roots and Φ^- be the corresponding set of negative roots. The set of simple roots determined by Φ^+ is $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. We will use throughout this paper the ordering of simple roots given in [Hum1] following Bourbaki. Given a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ which is a sum of root spaces, let $\Phi(\mathfrak{a})$ denote the corresponding set of roots. Let B be the Borel subgroup relative to (G, T) given by the set of negative roots and let U be the unipotent radical of B. More generally, if $J \subseteq \Delta$, let P_J be the parabolic subgroup relative to $-J$ and let U_J be the unipotent radical and L_J the Levi factor of P_J. Let Φ_J be the root subsystem in Φ generated by the simple roots in J, with positive subset $\Phi_J^+ = \Phi_J \cap \Phi^+$. Set $\mathfrak{g} = \text{Lie } G$, $\mathfrak{b} = \text{Lie } B$, $\mathfrak{u} = \text{Lie } U$, $\mathfrak{p}_J = \text{Lie } P_J$, $\mathfrak{l}_J = \text{Lie } L_J$, and $\mathfrak{u}_J = \text{Lie } U_J$.

Let $\mathbb{E}$ be the Euclidean space associated with Φ, and denote the inner product on $\mathbb{E}$ by $\langle\ ,\ \rangle$. Let $\check{\alpha}$ be the coroot corresponding to $\alpha \in \Phi$. Set α_0 to be the highest short root. Let ρ be the half sum of positive roots. The Coxeter number associated to Φ is $h = \langle \rho, \check{\alpha}_0 \rangle + 1$.

Let $X := X(T)$ be the integral weight lattice spanned by the fundamental weights $\{\omega_1, \ldots, \omega_l\}$. Let M be a finite-dimensional T-module and $M = \oplus_{\lambda \in X} M_\lambda$ be its weight space decomposition. The character of M, denoted by $\text{ch } M = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X(T)]$. If M and N are T-modules such that $\dim M_\lambda \leq \dim N_\lambda$ for all λ then we say that $\text{ch } M \leq \text{ch } N$. The set X has a partial ordering defined as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\, \alpha$. The set of dominant integral weights is denoted by $X^+ = X(T)_+$ and the set of p^r-restricted weights is $X_r = X_r(T)$. For $J \subseteq \Delta$, the set of J-dominant weights is

$$X_J^+ := \{\, \mu \in X \mid \langle \mu, \check{\alpha} \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Phi_J^+ \,\}.$$

and denote the p-restricted J-weights by $(X_J)_1$. The bottom alcove $\overline{C}_{\mathbb{Z}}$ is defined as

$$\overline{C}_{\mathbb{Z}} := \{\lambda \in X \mid 0 \leq \langle \lambda + \rho, \check{\alpha}_0 \rangle \leq p\}.$$

Set $H^0(\lambda) = \text{ind}_B^G \lambda$ where λ is the one-dimensional B-module obtained from the character $\lambda \in X^+$ by letting U act trivially. The Weyl group corresponding to Φ is W and acts on X via the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ where $w \in W$, $\lambda \in X$.

2. Cohomology and Composition Factors

2.1. For this section, let $R = \mathbb{Z}$, $\mathbb{C}$ or $\overline{\mathbb{F}}_p$, and let $J \subseteq \Delta$. Then $\mathfrak{u}_J$ has a basis consisting of root vectors where the structure constants are in R. In order to construct such a basis one can take an appropriate subset of the Chevalley basis for $\mathfrak{g}$. The standard complex on $\Lambda^\bullet(\mathfrak{u}_J^*)$ has differentials which are R-linear maps and we will denote the cohomology of this complex by $\text{H}^\bullet(\mathfrak{u}_J, R)$. Moreover, the torus T acts on the standard complex $\Lambda^\bullet(\mathfrak{u}_J^*)$. The differentials respect the T-action so it suffices to look at the smaller complexes $(\Lambda^\bullet(\mathfrak{u}_J^*))_\lambda$. The cohomology of this complex will be denoted by $\text{H}^\bullet(\mathfrak{u}_J, R)_\lambda$. For each n, $(\Lambda^n(\mathfrak{u}_J^*))_\lambda$ is a free R-module of finite rank, so the cohomology $\text{H}^n(\mathfrak{u}_J, R)_\lambda$ is a finitely generated R-module.

One can use the arguments given in Knapp [Kna, Theorem 6.10] to show that the cohomology groups when $R = \mathbb{C}$ or $\overline{\mathbb{F}}_p$ satisfy Poincaré Duality:

$$\mathrm{H}^n(\mathfrak{u}_J, R) \cong \mathrm{H}^{N-n}(\mathfrak{u}_J, R)^* \otimes \Lambda^N(\mathfrak{u}_J^*) \tag{2.1.1}$$

as T-modules where $N = \dim \mathfrak{u}_J$. The Universal Coefficient Theorem (UCT) (cf. [R, Theorem 8.26]) can be used to relate the cohomology over $\mathbb{Z}$ to the cohomology over $\mathbb{C}$ and $\overline{\mathbb{F}}_p$. The $\mathbb{Z}$-module $\mathbb{C}$ is divisible, so from the UCT (cf. [R, Corollary 8.28]) we have

$$\mathrm{H}^n(\mathfrak{u}_J, \mathbb{C})_\lambda \cong \mathrm{H}^n(\mathfrak{u}_J, \mathbb{Z})_\lambda \otimes_{\mathbb{Z}} \mathbb{C}. \tag{2.1.2}$$

On the other hand, when $k = \overline{\mathbb{F}}_p$, the UCT shows that

$$\mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)_\lambda \cong (\mathrm{H}^n(\mathfrak{u}_J, \mathbb{Z})_\lambda \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\overline{\mathbb{F}}_p, \mathrm{H}^{n-1}(\mathfrak{u}_J, \mathbb{Z})_\lambda). \tag{2.1.3}$$

For every n, the formulas (2.1.2) and (2.1.3) demonstrate that

$$\dim \mathrm{H}^n(\mathfrak{u}_J, \mathbb{C})_\lambda \leq \dim \mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)_\lambda.$$

In particular, ch $\mathrm{H}^n(\mathfrak{u}_J, \mathbb{C}) \leq$ ch $\mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$. One should observe that additional cohomology classes in $\mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)_\lambda$ can arise from either the first or second summand in (2.1.3) because of p-torsion in $\mathrm{H}^\bullet(\mathfrak{u}_J, \mathbb{Z})_\lambda$.

For a $\mathfrak{u}_J$-module, one can define $\mathrm{H}^n(\mathfrak{u}_J, M)$ using a complex involving $\Lambda^\bullet(\mathfrak{u}_J^*) \otimes M$ [Jan, I 9.17]. If M is a $\mathfrak{p}_J$-module then $\mathrm{H}^n(\mathfrak{u}_J, M)$ is a $\mathfrak{l}_J$-module. If M, N are arbitrary $\mathfrak{u}_J$-modules then $\mathrm{Ext}^n_{\mathfrak{u}}(M, N) = \mathrm{H}^n(\mathfrak{u}_J, M^* \otimes N)$ for $n \geq 0$.

2.2. Category $\mathcal{O}_J$. For this section, $k = \mathbb{C}$. Fix $J \subseteq \Delta$. Denote the Weyl group of Φ_J by W_J, viewed as a subgroup of W. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Definition 2.2.1. Let $\mathcal{O}_J$ be the full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules consisting of modules V which satisfy the following conditions:

(i) The module V is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.
(ii) As a $\mathcal{U}(\mathfrak{l}_J)$-module, V is the direct sum of finite-dimensional $\mathcal{U}(\mathfrak{l}_J)$-modules.
(iii) If $v \in V$, then $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{u}_J)v < \infty$.

Let Z be the center of $\mathcal{U}(\mathfrak{g})$ and denote the set of algebra homomorphisms $Z \to \mathbb{C}$ by $Z^\sharp$. We say that $\chi \in Z^\sharp$ is a *central character* of $V \in \mathcal{O}_J$ if $zv = \chi(z)v$ for all $z \in Z$ and all $v \in V$. For each $\chi \in Z^\sharp$, let $\mathcal{O}_J^\chi$ be the full subcategory of $\mathcal{O}_J$ consisting of modules $V \in \mathcal{O}_J$ such that for all $z \in Z$ and $v \in V$, v is annihilated by some power of $z - \chi(z)$. We have the decomposition

$$\mathcal{O}_J = \bigoplus_{\chi \in Z^\sharp} \mathcal{O}_J^\chi.$$

We call $\mathcal{O}_J^\chi$ an *infinitesimal block* of category $\mathcal{O}_J$.

For the purpose of this paper we will only need to apply information about the integral blocks so we can assume that the weights which arise are in X. The key objects in integral blocks of $\mathcal{O}_J$ are the parabolic Verma modules, which are defined as follows. For a finite-dimensional irreducible $\mathfrak{l}_J$-module $L_J(\mu)$ with highest weight $\mu \in X_J^+$ extend $L_J(\mu)$ to a $\mathfrak{p}_J$-module by letting $\mathfrak{u}_J^+$ act trivially. The induced module

$$Z_J(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_J)} L_J(\mu)$$

is a *parabolic Verma module*, which we will abbreviate as PVM.

The module $Z_J(\mu)$ has a unique maximal submodule and hence a unique simple quotient module, which we denote by $L(\mu)$; $L(\mu)$ is also the unique simple quotient of the ordinary Verma module $Z(\mu) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mu$. All simple modules in the integral blocks of $\mathcal{O}_J$ are isomorphic to some $L(\mu)$. For each $\mu \in X$, the ordinary Verma module $Z(\mu)$ (and any quotient thereof, such as $Z_J(\mu)$ or $L(\mu)$ if $\mu \in X_J^+$) has a central character which we will denote by $\chi_\mu \in Z^\sharp$. If $\chi = \chi_\mu$, write $\mathcal{O}_J^\mu := \mathcal{O}_J^{\chi_\mu}$. The Harish-Chandra linkage principle yields

$$\chi_\mu = \chi_\nu \quad \Leftrightarrow \quad \nu \in W \cdot \mu.$$

This implies that the simple modules (and hence the PVM's and projective indecomposable modules) in $\mathcal{O}_J^\mu$ can be indexed by $\{w \in W \mid w \cdot \mu \in X_J^+\}$ (by identifying repetitions).

For $\mu \in X$, let

$$\Phi_\mu = \{\alpha \in \Phi \mid \langle \mu + \rho, \check{\alpha} \rangle = 0\}.$$

If $\Phi_\mu = \varnothing$, then we say that μ is a *regular weight*; otherwise, it is a *singular weight*. If μ and ν are both regular weights, then $\mathcal{O}_J^\mu$ is equivalent to $\mathcal{O}_J^\nu$ by the Jantzen-Zuckerman translation principle.

For each $\alpha \in \Phi$, let $s_\alpha \in W$ denote the reflection in $\mathbb{E}$ about the hyperplane orthogonal to α. If μ is a regular dominant weight, then $\{w \in W \mid w \cdot \mu \in X_J^+\}$ is the set

$$(2.2.1) \qquad {}^J W = \{w \in W \mid l(s_\alpha w) = l(w) + 1 \text{ for all } \alpha \in J\} = \{w \in W \mid w^{-1}(\Phi_J^+) \subseteq \Phi^+\}$$

which is the set of minimal length right coset representatives of W_J in W. Let w_0 (resp. $w_J, {}^J w$) denote the longest element in W (resp. $W_J, {}^J W$). Then $w_0 = w_J {}^J w$.

2.3. The following theorem provides information about the L_J composition factors in $\mathrm{H}^\bullet(\mathfrak{u}_J, L(\mu))$ when $k = \mathbb{C}$. For V a finite dimensional semisimple L_J-module, write $[V : L_J(\sigma)]_{L_J}$ for the multiplicity of $L_J(\sigma)$ as an L_J-composition factor of V.

THEOREM 2.3.1. *Let $k = \mathbb{C}$, $V \in \mathcal{O}_J$ and $\lambda \in X$.*

(a) $\mathrm{Ext}^i_{\mathcal{O}_J}(Z_J(\lambda), V) \cong \mathrm{Hom}_{\mathfrak{l}_J}(L_J(\lambda), \mathrm{H}^i(\mathfrak{u}_J, V))$

(b) *If $[\mathrm{H}^i(\mathfrak{u}_J, L(\mu)) : L_J(\sigma)]_{L_J} \neq 0$ where $\mu \in X_+$ then $\sigma = w \cdot \mu$ where $w \in {}^J W$.*

PROOF. (a) First observe that $\mathrm{Ext}^i_{\mathcal{O}_J}(Z_J(\lambda), V) \cong \mathrm{Ext}^i_{(\mathfrak{g},\mathfrak{l}_J)}(Z_J(\lambda), V)$ (relative Lie algebra cohomology) and by Frobenius reciprocity we have

$$\mathrm{Ext}^i_{(\mathfrak{g},\mathfrak{l}_J)}(Z_J(\lambda), V) \cong \mathrm{Ext}^i_{(\mathfrak{p}_J,\mathfrak{l}_J)}(L_J(\lambda), V) \cong \mathrm{H}^i(\mathfrak{p}_J, \mathfrak{l}_J; L_J(\lambda)^* \otimes V).$$

Since $\mathfrak{u}_J \trianglelefteq \mathfrak{p}_J$, one can use the Grothendieck spectral sequence construction given in [Jan, I Proposition 4.1] to obtain a spectral sequence,

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{p}_J/\mathfrak{u}_J, \mathfrak{l}_J/(\mathfrak{l}_J \cap \mathfrak{u}_J)); \mathrm{H}^j(\mathfrak{u}_J, 0; L_J(\lambda)^* \otimes V) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{p}_J, \mathfrak{l}_J; L_J(\lambda)^* \otimes V).$$

However, $E_2^{i,j} \cong \mathrm{H}^i(\mathfrak{l}_J, \mathfrak{l}_J; \mathrm{H}^j(\mathfrak{u}_J, 0; L_J(\lambda)^* \otimes V)) = 0$ for $i > 0$, so the spectral sequence collapses and yields

$$\mathrm{Hom}_{\mathfrak{l}_J}(L_J(\lambda), \mathrm{H}^j(\mathfrak{u}_J, V)) \cong \mathrm{H}^0(\mathfrak{l}_J, \mathfrak{l}_J; \mathrm{H}^j(\mathfrak{u}_J, L_J(\lambda)^* \otimes V)) \cong \mathrm{H}^j(\mathfrak{p}_J, \mathfrak{l}_J; L_J(\lambda)^* \otimes V).$$

(b) Suppose that $[\mathrm{H}^i(\mathfrak{u}_J, L(\mu)) : L_J(\sigma)]_{L_J} \neq 0$. Then from part (a),

$$[\mathrm{H}^i(\mathfrak{u}_J, L(\mu)) : L_J(\sigma)]_{L_J} = \dim \mathrm{Hom}_{\mathfrak{l}_J}(L_J(\sigma), \mathrm{H}^i(\mathfrak{u}_J, L(\mu))) = \dim \mathrm{Ext}^i_{\mathcal{O}_J}(Z_J(\sigma), L(\mu)).$$

But, $\mathrm{Ext}^i_{\mathcal{O}_J}(Z_J(\sigma), L(\mu)) \neq 0$ implies by linkage that $\sigma = w \cdot \mu$ where $w \in {}^J W$. $\square$

2.4. Now let us assume that $k=\overline{\mathbb{F}}_p$. Let W_p be the affine Weyl group and $\widehat{W}_p$ be the extended affine Weyl group. In this setting we regard G as an affine reductive group scheme with $F:G\to G$ denoting the Frobenius morphism. Let F^r be this morphism composed with itself r times and set $G_rT=(F_r)^{-1}(T)$. The category of G_rT-modules has a well developed representation theory (cf. [Jan, II Chapter 9]). Group schemes analogous to G_rT can be defined similarly using the Frobenius morphism for L_J, P_J, B, U, etc.

The following theorem provides information about the composition factors in the $\mathfrak{u}_J$-cohomology for $p\geq 3$.

THEOREM 2.4.1. *Let $k=\mathbb{F}_p$ with $p\geq 3$.*

(a) *If $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu)):L_J(\sigma)]_{L_J}\neq 0$ where $\mu\in X^+$ then $\mu=w\cdot\sigma$ where $w\in\widehat{W}_p$.*
(b) *If $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu)):L_J(\sigma)]_{L_J}\neq 0$ where $\mu\in X_1$ and $\sigma\in(X_J)_1$ then $\mu=w\cdot\sigma$ where $w\in W_p$.*

PROOF. (a) Suppose that $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu)):L_J(\sigma)]_{L_J}\neq 0$. From the Steinberg tensor product theorem, we can write $L_J(\sigma)=L_J(\sigma_0)\otimes L_J(\sigma_1)^{(1)}$ where $\sigma_0\in(X_J)_1$ and $\sigma_1\in X_J^+$. Therefore, $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu)):L_J(\sigma_0)\otimes p\gamma_1]_{(L_J)_1T}\neq 0$ for some $\gamma_1\in X$. One can also express $\mu=\mu_0+p\mu_1$ where $\mu_0\in X_1$ and $\mu_1\in X^+$ so that

$$\mathrm{H}^i(\mathfrak{u}_J,L(\mu))\cong\mathrm{H}^i(\mathfrak{u}_J,L(\mu_0))\otimes L(\mu_1)^{(1)}.$$

Therefore, $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu)):L_J(\sigma_0)\otimes p\gamma_1]_{(L_J)_1T}\neq 0$ implies that $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu_0))\otimes p\gamma_2 : L_J(\sigma_0)\otimes p\gamma_1]_{(L_J)_1T}\neq 0$ for some $\gamma_2\in X$, thus $[\mathrm{H}^i(\mathfrak{u}_J,L(\mu_0)):L_J(\sigma_0)\otimes p\gamma]_{(L_J)_1T}\neq 0$ for some $\gamma\in X$ (where $\gamma=\gamma_1-\gamma_2$).

Observe that

$$[\mathrm{H}^i(\mathfrak{u}_J,L(\mu_0)):L_J(\sigma_0)\otimes p\gamma]_{(L_J)_1T}=\dim\operatorname{Hom}_{(L_J)_1T}(P_J(\sigma_0)\otimes p\gamma,\mathrm{H}^i(\mathfrak{u}_J,L(\mu_0))).\tag{2.4.1}$$

where $P_J(\sigma_0)\otimes p\gamma$ is the $(L_J)_1T$ projective cover of $L_J(\sigma_0)\otimes p\gamma$.

Next consider the composition factor multiplicities for the cohomology of $L(\mu_0)$ over the Frobenius kernel $(U_J)_1$,

$$[\mathrm{H}^i((U_J)_1,L(\mu_0)):L_J(\sigma_0)\otimes p\gamma]_{(L_J)_1T}=\dim\operatorname{Hom}_{(L_J)_1T}(P_J(\sigma_0)\otimes p\gamma,\mathrm{H}^i((U_J)_1,L(\mu_0))).$$

We can also give another interpretation of this composition factor multiplicity. First, let us apply the Lyndon-Hochschild-Serre spectral sequence for $(U_J)_1\trianglelefteq(P_J)_1T$, $(P_J)_1T/(U_J)_1\cong(L_J)_1T$:

$$E_2^{i,j}=\operatorname{Ext}^i_{(L_J)_1T}(P_J(\sigma_0)\otimes p\gamma,\mathrm{H}^j((U_J)_1,L(\mu_0)))\Rightarrow\operatorname{Ext}^{i+j}_{(P_J)_1T}(P_J(\sigma_0)\otimes p\gamma,L(\mu_0)).\tag{2.4.2}$$

Since $P:=P_J(\sigma_0)\otimes p\gamma$ is projective as an $(L_J)_1T$-module, the spectral sequence collapses and we have

$$\begin{aligned}\operatorname{Hom}_{(L_J)_1T}(P,\mathrm{H}^i((U_J)_1,L(\mu_0)) &\cong \operatorname{Ext}^i_{(P_J)_1T}(P,L(\mu_0))\\ &\cong \operatorname{Ext}^i_{G_1T}(\operatorname{coind}_{(P_J)_1T}^{G_1T}P,L(\mu_0)).\end{aligned}$$

For $p\geq 3$, there exists another first quadrant spectral sequence which can be used to relate these two different composition factor multiplicities [FP2, (1.3) Proposition]:

$$E_2^{2i,j}=S^i(\mathfrak{u}_J^*)^{(1)}\otimes\mathrm{H}^j(\mathfrak{u}_J,L(\mu_0))\Rightarrow\mathrm{H}^{2i+j}((U_J)_1,L(\mu_0)).$$

Since the functor $\mathrm{Hom}_{(L_J)_1T}(P,-)$ is exact, we can compose it with the spectral sequence above to get another spectral sequence:
(2.4.3)
$$E_2^{2i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes \mathrm{Hom}_{(L_J)_1T}(P, \mathrm{H}^j(\mathfrak{u}_J, L(\mu_0))) \Rightarrow \mathrm{Hom}_{(L_J)_1T}(P, \mathrm{H}^{2i+j}((U_J)_1, L(\mu_0))).$$

Suppose that $\sigma_0 + p\gamma \notin W_p \cdot \mu_0$. Then by the linkage principle for G_1T:
$$\mathrm{Hom}_{(L_J)_1T}(P_J(\sigma_0)\otimes p\gamma, \mathrm{H}^i((U_J)_1, L(\mu_0))) \cong \mathrm{Ext}^i_{G_1T}(\mathrm{coind}_{(P_J)_1T}^{G_1T} P_J(\sigma_0)\otimes p\gamma, L(\mu_0)) = 0$$
for all $i \geq 0$. Therefore, the spectral sequence (2.4.3) abuts to zero. The differential d_2 in the spectral sequence maps $E_2^{0,j}$ to $E_2^{2,j-1}$. Note that $E_2^{2i,j} = S^i(\mathfrak{u}_J^*)^{(1)} \otimes E_2^{0,j}$ for all $i,j \geq 0$. Since $0 = E_0 = E_2^{0,0}$, it follows that $E_2^{2i,0} = 0$ for $i \geq 0$. Therefore, $E_2^{0,1} = 0$, thus $E_2^{2i,1} = 0$ for $i \geq 0$. Continuing in this fashion, we have $E_2^{2i,j} = 0$ for all i,j. In particular, using (2.4.1) and (2.4.3), $[\mathrm{H}^j(\mathfrak{u}_J, L(\mu_0)) : L_J(\sigma_0)\otimes p\gamma]_{(L_J)_1T} = \dim \mathrm{Hom}_{(L_J)_1T}(P, \mathrm{H}^j(\mathfrak{u}_J, L(\mu_0))) = \dim E_2^{0,j} = 0$ for all j which is a contradiction. This implies that μ_0 and σ_0 are in the same orbit under $\widehat{W}_p$, thus $\mu = w\cdot\sigma$ where $w \in \widehat{W}_p$.

(b) Under the hypotheses, we can apply the above argument with $0 = \gamma_1 = \gamma_2 = \gamma$. Therefore, $\mu = w\cdot\sigma$ where $w \in W_p$. □

2.5. We present the following proposition which allows one to compare composition factors of the cohomology with coefficients in a module to the cohomology with trivial coefficients. Note that this proposition is independent of the characteristic of the field k.

Proposition 2.5.1. *Let $J \subseteq \Delta$ and V be a finite-dimensional P_J-module. If $[\mathrm{H}^i(\mathfrak{u}_J, V) : L_J(\sigma)]_{L_J} \neq 0$ for $\sigma \in X_J^+$ then $[\mathrm{H}^i(\mathfrak{u}_J, k)\otimes V : L_J(\sigma)]_{L_J} \neq 0$.*

Proof. The simple finite-dimensional P_J-modules are the simple finite-dimensional L_J-modules inflated to P_J by making U_J act trivially. We will prove the proposition by induction on the composition length n of V. For $n = 1$, this is clear because V is simple and U_J acts trivially so
$$\mathrm{H}^i(\mathfrak{u}_J, V) \cong \mathrm{H}^i(\mathfrak{u}_J, k)\otimes V.$$

Now assume that the proposition holds for modules of composition length n, and let V have composition length $n+1$. There exists a short exact sequence
$$0 \to V' \to V \to L \to 0$$
where V' has composition length n and L is a simple P_J-module. We have a long exact sequence in cohomology which shows that if $[\mathrm{H}^i(\mathfrak{u}_J, V) : L_J(\sigma)]_{L_J} \neq 0$ then either $[\mathrm{H}^i(\mathfrak{u}_J, V') : L_J(\sigma)]_{L_J} \neq 0$ or $[\mathrm{H}^i(\mathfrak{u}_J, L) : L_J(\sigma)]_{L_J} \neq 0$. By the induction hypothesis, this implies $[\mathrm{H}^i(\mathfrak{u}_J, k)\otimes V' : L_J(\sigma)]_{L_J} \neq 0$ or $[\mathrm{H}^i(\mathfrak{u}_J, k)\otimes L : L_J(\sigma)]_{L_J} \neq 0$.

The short exact sequence above can be tensored by $\mathrm{H}^i(\mathfrak{u}_J, k)$ to obtain a short exact sequence:
$$0 \to \mathrm{H}^i(\mathfrak{u}_J, k)\otimes V' \to \mathrm{H}^i(\mathfrak{u}_J, k)\otimes V \to \mathrm{H}^i(\mathfrak{u}_J, k)\otimes L \to 0.$$
The result now follows because one of the terms on the end has an L_J composition factor of the form $L_J(\sigma)$ by the induction hypothesis, so the middle term has to have a composition factor of this form. □

3. Parabolic Computations

In this section we prove several elementary results which will be ingredients in our proof of Kostant's Theorem and its generalization to prime characteristic in Section 4.

3.1. Given $\Psi \subset \Phi^+$, write

$$\langle \Psi \rangle = \sum_{\beta \in \Psi} \beta.$$

For $w \in W$ put

$$\Phi(w) = -(w\Phi^+ \cap \Phi^-) = w\Phi^- \cap \Phi^+ \subset \Phi^+. \tag{3.1.1}$$

We recall some basic facts about $\Phi(w)$.

Lemma 3.1.1. *Let $w \in W$.*

(a) $|\Phi(w)| = l(w)$.
(b) $w \cdot 0 = -\langle \Phi(w) \rangle$.
(c) *If $w = s_{j_1} \dots s_{j_t}$ is a reduced expression, then*

$$\Phi(w^{-1}) = \{\alpha_{j_t}, s_{j_t}\alpha_{j_{t-1}}, s_{j_t}s_{j_{t-1}}\alpha_{j_{t-2}}, \dots, s_{j_t} \dots s_{j_2}\alpha_{j_1}\}.$$

Proof. (a) [Hum1, Lemma 10.3A], (b) [Kna, Proposition 3.19], (c) [Hum2, Exercise 5.6.1] □

Lemma 3.1.2. *Let $J \subseteq \Delta$ and $w \in W$.*

(a) *$\Phi(w) \subset \Phi^+ \setminus \Phi_J^+ = \Phi(\mathfrak{u}_J)$ if and only if $w \in {}^JW$.*
(b) *If $w \cdot 0 = -\langle \Psi \rangle$ for some $\Psi \subset \Phi^+$ then $\Psi = \Phi(w)$.*

Proof. (a) Assume $w \in {}^JW$. Let $\beta \in \Phi(w)$. Then $\beta \in \Phi^+$, and $\beta \in w\Phi^-$ whence $w^{-1}\beta \in \Phi^-$. Thus $\beta \notin \Phi_J^+$ by the second characterization of ${}^JW$ in (2.2.1).

Conversely, assume $w \notin {}^JW$. Then by the first characterization of ${}^JW$ in (2.2.1), w has a reduced expression beginning with s_α for some $\alpha \in J$ (by the Exchange Condition, for instance). Then by Lemma 3.1.1(c), $\alpha \in \Phi(w)$; but $\alpha \in \Phi_J^+$ so $\Phi(w) \not\subset \Phi^+ \setminus \Phi_J^+$.

(b) We prove this by induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and $w \cdot 0 = 0$, so clearly the only possible Ψ is $\Psi = \varnothing = \Phi(1)$.

Given w with $l(w) > 0$, write $w = s_\alpha w'$ with $\alpha \in \Delta$ and $l(w') = l(w) - 1$. Then $\alpha \in \Phi(w)$ and $\alpha \notin \Phi(w') = s_\alpha(\Phi(w) \setminus \{\alpha\})$; cf. the proof of [Hum2, Lemma 1.6]. Suppose $w \cdot 0 = -(\gamma_1 + \dots + \gamma_m)$ for distinct $\gamma_1, \dots, \gamma_m \in \Phi^+$. Then

$$w' \cdot 0 = s_\alpha \cdot (w \cdot 0) = s_\alpha(w \cdot 0) + s_\alpha\rho - \rho = -(s_\alpha\gamma_1 + \dots + s_\alpha\gamma_m + \alpha).$$

There are two cases.

Case 1: No $\gamma_i = \alpha$. Then $s_\alpha\gamma_1, \dots, s_\alpha\gamma_m, \alpha$ are distinct positive roots: s_α permutes the positive roots other than α, and no $s_\alpha\gamma_i = \alpha$ because $s_\alpha(-\alpha) = \alpha$. But then by induction, $\{s_\alpha\gamma_1, \dots, s_\alpha\gamma_m, \alpha\} = \Phi(w')$, and this contradicts $\alpha \notin \Phi(w')$.

Case 2: Some $\gamma_i = \alpha$. Say $\gamma_m = \alpha$. Then $s_\alpha(\gamma_m) = -\alpha$, so $w' \cdot 0 = -(s_\alpha\gamma_1 + \dots + s_\alpha\gamma_{m-1})$. By induction, $\Phi(w') = \{s_\alpha\gamma_1, \dots, s_\alpha\gamma_{m-1}\}$. Hence $\Phi(w) = s_\alpha\Phi(w') \cup \{\alpha\} = \{\gamma_1, \dots, \gamma_m\}$ as required. □

3.2. Saturation. Lemma 3.1.2 guarantees that, for $w \in {}^JW$, $w \cdot 0 = -\langle \Phi(w) \rangle$ is a weight in $\Lambda^n(\mathfrak{u}_J^*)$, where $n = l(w)$. Specifically, if $\Phi(w) = \{\beta_1, \ldots, \beta_n\}$ then the vector $f_{\Phi(w)} := f_{\beta_1} \wedge \cdots \wedge f_{\beta_n}$ has the desired weight, where $\{f_\beta \mid \beta \in \Phi(\mathfrak{u}_J)\}$ is the basis for $\mathfrak{u}_J^*$ dual to a fixed basis of weight vectors $\{x_\beta \mid \beta \in \Phi(\mathfrak{u}_J)\}$ for $\mathfrak{u}_J$. Lemmas 3.1.1 and 3.1.2 guarantee that the weight $w \cdot 0$ occurs with multiplicity one in $\Lambda^\bullet(\mathfrak{u}_J^*)$. In particular, since the differentials in the complex $0 \to \Lambda^\bullet(\mathfrak{u}_J^*)$ preserve weights, we see that $f_{\Phi(w)}$ descends to an element of $\mathrm{H}^n(\mathfrak{u}_J, k)$ of weight $w \cdot 0$, and n is the only degree in which this weight occurs in $\mathrm{H}^\bullet(\mathfrak{u}_J, k)$ (where $k = \mathbb{C}$ or $\overline{\mathbb{F}}_p$).

In order to prove that $f_{\Phi(w)}$ generates an L_J-submodule of $\mathrm{H}^\bullet(\mathfrak{u}_J, k)$ of highest weight $w \cdot 0$, we need the following condition, which could be described by saying that $\Phi(w)$ is "saturated" with respect to Φ_J^+.

Proposition 3.2.1. *Let $w \in {}^JW$. If $\beta \in \Phi(w)$, $\gamma \in \Phi_J^+$, and $\delta = \beta - \gamma \in \Phi$, then $\delta \in \Phi(w)$.*

Proof. We prove this by induction on $l(w)$. If $w = 1$ then $\Phi(w) = \varnothing$ and the statement is vacuously true. So assume $l(w) > 0$. Write $w = w's_\alpha$ with $\alpha \in \Delta$ and $l(w') = l(w) - 1$; then necessarily $w' \in {}^JW$. To see this, note that $w\alpha < 0$, so $(w')^{-1}(\Phi_J^+) = s_\alpha w^{-1}(\Phi_J^+) \subset s_\alpha(\Phi^+ \smallsetminus \{\alpha\}) \subset \Phi^+$. Now

$$\begin{aligned} \Phi(w) &= \Phi^+ \cap w's_\alpha\Phi^- \\ &= \Phi^+ \cap w'(\Phi^- \smallsetminus \{-\alpha\} \cup \{\alpha\}) \\ &= (\Phi^+ \cap w'\Phi^-) \cup \{w'\alpha\} \\ &= \Phi(w') \cup \{w'\alpha\}, \end{aligned}$$

where in the third equality we have used the fact that $w'\alpha > 0$. By induction, $\Phi(w')$ is saturated with respect to Φ_J^+. So it remains to check the condition of the lemma when $\beta = w'\alpha$.

Let $\beta = w'\alpha$ and suppose $\delta = \beta - \gamma \in \Phi$ for some $\gamma \in \Phi_J^+$. Since $\beta \in \Phi^+ \smallsetminus \Phi_J^+$ by Lemma 3.1.2, and $\gamma \in \Phi_J^+$, necessarily $\delta \in \Phi^+ \smallsetminus \Phi_J^+$. Consider the root $(w')^{-1}\delta = (w')^{-1}(w'\alpha - \gamma) = \alpha - (w')^{-1}(\gamma)$. Since $w' \in {}^JW$ and $\gamma \in \Phi_J^+$, we know $(w')^{-1}(\gamma) > 0$. Since α is simple, $(w')^{-1}\delta < 0$. That is, $\delta \in w'(\Phi^-)$. Thus, $\delta \in \Phi(w') \subset \Phi(w)$, as required. □

3.3. Prime characteristic. In the prime characteristic setting we will need to work harder than in characteristic zero, because our control over the composition factors in cohomology in Theorem 2.4.1 is much weaker than in Theorem 2.3.1. We begin by recording two simple technical facts which will be needed later.

Proposition 3.3.1. (a) *Let $\lambda, \mu \in X$ and suppose $\lambda = w\mu$ where $w = s_{j_1} \ldots s_{j_t}$ with t minimal. Then $\langle \alpha_{j_r}, s_{j_{r+1}} \ldots s_{j_t}\mu \rangle \neq 0$ for $1 \leq r \leq t-1$.*
(b) *Suppose $\widetilde{\alpha} \in \Phi^+$ has maximal height in its W-orbit. Then $\langle \beta, \widetilde{\alpha} \rangle \geq 0$ for all $\beta \in \Phi^+$.*

Proof. (a) Since t is minimal,

$$s_{j_{r+1}} \ldots s_{j_t}\mu \neq s_{j_r} \ldots s_{j_t}\mu = s_{j_{r+1}} \ldots s_{j_t}\mu - \langle s_{j_{r+1}} \ldots s_{j_t}\mu, \check{\alpha}_{j_r} \rangle \alpha_{j_r}.$$

This implies the desired inequality.

(b) Otherwise, $s_\beta(\widetilde{\alpha}) = \widetilde{\alpha} - \langle \widetilde{\alpha}, \check{\beta} \rangle \beta$ would be a root of the same length as $\widetilde{\alpha}$, but higher, contradicting the hypothesis. □

We will be able to cut down the possible weights in cohomology when $p \geq h-1$. The proof will make use of certain special sums of positive roots. For $1 \leq i \leq l$ set

$$\begin{aligned} \Phi_i &= \{\, \alpha \in \Phi^+ \mid \langle \omega_i, \alpha \rangle > 0 \,\} \\ &= \{\, \alpha \in \Phi^+ \mid \alpha = \sum r_j \alpha_j \text{ with } r_i > 0 \,\} \\ &= \Phi^+ \smallsetminus \Phi_J^+, \text{ where} \\ J = J_i &= \Delta \smallsetminus \{\alpha_i\}, \\ \Phi_i' &= \{\, \alpha \in \Phi_i \mid \langle \alpha, \alpha_i \rangle \geq 0 \,\}, \end{aligned} \tag{3.3.1}$$

and define

$$\delta_i = \langle \Phi_i \rangle, \qquad \delta_i' = \langle \Phi_i' \rangle. \tag{3.3.2}$$

We begin by collecting some elementary properties of δ_i.

Proposition 3.3.2. (a) *$w(\Phi_i) = \Phi_i$ for all $w \in W_J$.*
(b) *$\delta_i = c\omega_i$ for some $c \in \mathbb{Z}$.*
(c) *$-\delta_i = {}^J w \cdot 0$ where $J = \Delta \smallsetminus \{\alpha_i\}$ (recall ${}^J w$ is the longest element of ${}^J W$).*

Proof. (a) For $j \neq i$ and α a positive root involving α_i, $s_j(\alpha)$ is again a positive root involving α_i. Thus s_j permutes Φ_i. Since the s_j with $j \neq i$ generate W_J, the result follows.

(b) From (a), for $j \neq i$, $s_j(\delta_i) = \delta_i$. Thus when δ_i is written as a linear combination of fundamental dominant weights, the coefficient of ω_j is 0. That is, $\delta_i = c\omega_i$ for some scalar c. Since $\delta_i \in \mathbb{Z}\Phi$, $c \in \mathbb{Z}$.

(c) Write

$$2\rho = \sum_{\substack{\alpha \in \Phi^+ \\ \langle \omega_i, \alpha \rangle > 0}} \alpha + \sum_{\substack{\alpha \in \Phi^+ \\ \langle \omega_i, \alpha \rangle = 0}} \alpha = \delta_i + 2\rho_J.$$

Apply the longest element w_J of W_J, and use the first computation in (a):

$$2 w_J \rho = w_J \delta_i - 2\rho_J = \delta_i - 2\rho_J.$$

Thus

$$w_J \rho = \tfrac{1}{2}\delta_i - \rho_J = \tfrac{1}{2}\delta_i - (\rho - \tfrac{1}{2}\delta_i) = \delta_i - \rho,$$

and so

$$-\delta_i = -w_J \rho - \rho = w_J w_0 \rho - \rho = {}^J w \cdot \rho.$$

□

3.4. The crucial property of δ_i' is that $\langle \delta_i', \check{\alpha}_i \rangle \leq h$. The proof will require a few steps. First, put $J = \Delta \smallsetminus \{\alpha_i\}$ as before, and recall that w_J denotes the longest element of the parabolic subgroup W_J. Let $w_i \in W$ be an element of shortest possible length such that

$$w_i w_J \alpha_i = \widetilde{\alpha}, \quad \text{the highest root in } W\alpha_i. \tag{3.4.1}$$

Proposition 3.4.1. *Let i, J, w_i and $\widetilde{\alpha}$ be as above.*
(a) *$w_J(\Phi_i \smallsetminus \Phi_i') = \Phi(w_i^{-1})$.*
(b) *$w_J(\delta_i - \delta_i') = \rho - w_i^{-1}\rho$.*

(c) $\langle \delta_i', \check{\alpha}_i \rangle = 1 + \langle \rho, \widetilde{\alpha}^\vee \rangle$.
(d) $\langle \delta_i', \check{\alpha}_i \rangle \leq h$.

PROOF. (a) Observe that $\beta \in w_J(\Phi_i \smallsetminus \Phi_i')$ if and only if $\beta = w_J\alpha$ with $\alpha \in \Phi_i$ and $\langle \alpha, \alpha_i \rangle < 0$; equivalently (using Proposition 3.3.2(a)), $\langle \beta, w_J\alpha_i \rangle < 0$ and $\beta \in \Phi_i$. Thus

$$\beta \in w_J(\Phi_i \smallsetminus \Phi_i') \iff \beta \in \Phi_i \text{ and } \langle w_i\beta, \widetilde{\alpha} \rangle < 0. \tag{3.4.2}$$

Assuming $\beta \in w_J(\Phi_i \smallsetminus \Phi_i')$, then $\beta \in \Phi^+$ and $w_i\beta \in \Phi^-$ (by Proposition 3.3.1(b)); equivalently $\beta \in \Phi(w_i^{-1})$ (by (3.1.1)).

To prove the reverse inclusion, assume that $\beta \in \Phi(w_i^{-1})$; i.e., $\beta \in \Phi^+$ and $w_i\beta \in \Phi^-$. We claim it is enough to show that $\langle w_i\beta, \widetilde{\alpha} \rangle < 0$ (the second condition of (3.4.2)). For if $\beta \notin \Phi_i$ then $\beta \in \Phi_J^+$, hence $w_J\beta \in \Phi_J^-$, and thus $\langle w_i\beta, \widetilde{\alpha} \rangle = \langle \beta, w_J\alpha_i \rangle = \langle w_J\beta, \alpha_i \rangle \geq 0$, since $\langle \alpha_j, \alpha_i \rangle \leq 0$ for $j \neq i$.

It remains to show $\langle w_i\beta, \widetilde{\alpha} \rangle < 0$, or, equivalently, $\langle w_i\beta, \widetilde{\alpha} \rangle \neq 0$, since $w_i\beta \in \Phi^-$ (recall Proposition 3.3.1(b)). Write $w_i = s_{j_1} \dots s_{j_t}$ with t minimal. By Lemma 3.1.1(c) we have $\beta = s_{j_t} \dots s_{j_{r+1}}\alpha_{j_r}$ for some $1 \leq r \leq t$. Put $\mu = w_J\alpha_i$ and $\lambda = \widetilde{\alpha}$ in Proposition 3.3.1(a) to obtain

$$\langle w_i\beta, \widetilde{\alpha} \rangle = \langle s_{j_1} \dots s_{j_r}\alpha_{j_r}, s_{j_1} \dots s_{j_t}w_J\alpha_i \rangle = \langle \alpha_{j_r}, s_{j_{r+1}} \dots s_{j_t}w_J\alpha_i \rangle \neq 0.$$

(b) Using (a) and Lemma 3.1.1(b),

$$w_J(\delta_i - \delta_i') = \langle w_J(\Phi_i \smallsetminus \Phi_i') \rangle = \langle \Phi(w_i^{-1}) \rangle = -w_i^{-1} \cdot 0 = \rho - w_i^{-1}\rho.$$

(c) Using (b) and the idea of the proof of Proposition 3.3.2(c),

$$\begin{aligned} \delta_i - \delta_i' = w_J(\rho - w_i^{-1}\rho) &= w_J[(\rho - \tfrac{1}{2}\delta_i) + \tfrac{1}{2}\delta_i] - w_Jw_i^{-1}\rho \\ &= -(\rho - \tfrac{1}{2}\delta_i) + \tfrac{1}{2}\delta_i - w_Jw_i^{-1}\rho = \delta_i - \rho - w_Jw_i^{-1}\rho. \end{aligned}$$

Thus $\delta_i' = \rho + w_Jw_i^{-1}\rho$ and so

$$\langle \delta_i', \check{\alpha}_i \rangle = \langle \rho + w_Jw_i^{-1}\rho, \check{\alpha}_i \rangle = 1 + \langle \rho, w_iw_J\check{\alpha}_i \rangle = 1 + \langle \rho, \widetilde{\alpha}^\vee \rangle.$$

(d) Combine (c) with the inequality $\langle \rho, \widetilde{\alpha}^\vee \rangle \leq \langle \rho, \check{\alpha}_0 \rangle = h - 1$. □

3.5. The next proposition is the key to our proof of Kostant's Theorem in characteristic $p \geq h - 1$.

PROPOSITION 3.5.1. *Assume $p \geq h - 1$. Suppose $\sigma = w \cdot 0 + p\mu$ is a weight of $\Lambda^\bullet(\mathfrak{u}^*)$ where $w \in W$ and $\mu \in X$. Then $\sigma = x \cdot 0$ for some $x \in W$.*

PROOF. The proof is again by induction on $l(w)$. Assume $w = 1$ so that $p\mu$ is a sum of distinct negative roots. Set $\nu = -\mu$ so that $p\nu = \langle \Psi \rangle$ for some $\Psi \subset \Phi^+$. For any $1 \leq i \leq l$,

$$\langle \langle \Psi \rangle, \check{\alpha}_i \rangle \leq \langle \delta_i, \check{\alpha}_i \rangle \leq \langle \delta_i', \check{\alpha}_i \rangle.$$

The first inequality follows because $\langle \alpha_j, \check{\alpha}_i \rangle \leq 0$ if $j \neq i$ whereas $\langle \alpha_i, \check{\alpha}_i \rangle = 2$, so including only positive roots that involve α_i can only make the inner product bigger. The second inequality follows similarly: including only those positive roots α with $\langle \alpha, \check{\alpha}_i \rangle \geq 0$ obviously can only increase the inner product. Writing $\langle\Psi\rangle = 2\rho - \langle\Psi^c\rangle$, where $\Psi^c = \Phi^+ \setminus \Psi$, applying the same inequality for Ψ^c, and using the fact that $\langle\rho, \check\alpha_i\rangle = 1$, we obtain

$$2 - \langle\delta_i', \check\alpha_i\rangle \le \langle\langle\Psi\rangle, \check\alpha_i\rangle \le \langle\delta_i', \check\alpha_i\rangle.$$

But we also have $\langle\delta_i', \check\alpha_i\rangle \le h$ by Proposition 3.4.1(d). Thus

$$2 - h \le p\langle\nu, \check\alpha_i\rangle \le h. \tag{3.5.1}$$

Since $p \ge h-1$ and $\langle\nu, \check\alpha_i\rangle \in \mathbb{Z}$, the first inequality implies $\langle\nu, \check\alpha_i\rangle \ge 0$ for all i. That is, ν is dominant. If $p > h$, the second inequality implies that $\langle\nu, \check\alpha_i\rangle = 0$ for all i, and thus $\nu = 0$. This completes the proof in the case $w = 1$ when $p > h$.

From Proposition 3.3.1(b), it follows that

$$p\langle\nu, \check\alpha_0\rangle = \langle\langle\Psi\rangle, \check\alpha_0\rangle \le \langle 2\rho, \check\alpha_0\rangle = 2(h-1).$$

Since $p \ge h-1$, we deduce that $\langle\nu, \check\alpha_0\rangle = 0,\ 1$ or 2. Suppose for the moment that we handle the case $\langle\nu, \check\alpha_0\rangle = 2$; this case does not arise if $p = h$. Recall also that we know ν is dominant. If $\langle\nu, \check\alpha_0\rangle = 0$ then $\nu = 0$; this can be seen since $\check\alpha_0$ is the highest root of the dual root system, and thus involves every dual simple root $\check\alpha_i$ with positive coefficient [Hum1, Lemma 10.4A]. So the coefficient of ω_i in ν must be 0 for every i. Suppose $\langle\nu, \check\alpha_0\rangle = 1$. Then ν is a minuscule dominant weight. Also $p\nu = \langle\Psi\rangle$ must belong to the root lattice.

When $p = h-1$, one can check for each irreducible root system that p does not divide the index of connection f (the index of the weight lattice in the root lattice); cf. [Hum1, p. 68]. Thus ν itself must lie in the root lattice. However, a case-by-case check using the list of minuscule weights (e.g., [Hum1, Exercise 13.13 and Table 13.1]) shows that this never happens.

Assume $p = h$. The Coxeter number is prime only in type A_l. In this case every fundamental dominant weight ω_i is minuscule, and $h = f = l+1$ so $p\omega_i$ is in the root lattice. Suppose $\nu = \omega_i$. Recall from Proposition 3.3.2(b) that $\delta_i = c\omega_i$; we compute

$$c = \langle c\omega_i, \check\alpha_i\rangle = \left\langle \sum_{\substack{\alpha\in\Phi^+ \\ \langle\omega_i,\alpha\rangle>0}} \alpha, \check\alpha_i \right\rangle = 2 + (l-1) = l+1 = h,$$

where we have used the fact that $\langle\alpha_i, \check\alpha_i\rangle = 2$, $\langle\alpha_j + \cdots + \alpha_i, \check\alpha_i\rangle = \langle\alpha_i + \cdots + \alpha_k, \check\alpha_i\rangle = 1$ for $1 \le j < i$ and $i < k \le l$, and $\langle\alpha, \check\alpha_i\rangle = 0$ for all other positive roots in type A_l which involve α_i. Thus $p\mu = -h\omega_i = -\delta_i = x\cdot 0$ for some $x \in W$ by Proposition 3.3.2(c), as required.

To complete the proof for $w = 1$, there remains to handle the case $\langle\nu, \check\alpha_0\rangle = 2$ when $p = h-1$. Set $\Psi_0 = \{\, \alpha\in\Phi^+ \mid \langle\alpha, \check\alpha_0\rangle > 0 \,\}$ and $\gamma = \langle\Psi_0\rangle$. We claim that $\gamma = (h-1)\alpha_0$. To see this, note that $s_{\alpha_0}\Psi_0 = -\Psi_0$ (recall that $\langle\alpha, \check\alpha_0\rangle \ge 0$ for $\alpha \in \Phi^+$). So $s_{\alpha_0}\gamma = -\gamma$. Substituting this into the formula for $s_{\alpha_0}\gamma$ gives $\gamma = \frac{1}{2}\langle\gamma, \check\alpha_0\rangle\alpha_0$. But $\langle\gamma, \check\alpha_0\rangle = \langle 2\rho, \check\alpha_0\rangle = 2(h-1)$, and this proves the claim.

Now assume $p = h-1$, $\langle\nu, \check\alpha_0\rangle = 2$, and $(h-1)\nu = \langle\Psi\rangle$ for some $\Psi \subset \Phi^+$. Then

$$2(h-1) = (h-1)\langle\nu, \check\alpha_0\rangle = \langle\langle\Psi\rangle, \check\alpha_0\rangle \le \langle 2\rho, \check\alpha_0\rangle = 2(h-1),$$

so we must have equality at the third step. It follows from the definition of Ψ_0 above, and the fact that $\langle\gamma, \check\alpha_0\rangle = 2(h-1)$, that $\Psi_0 \subset \Psi$. But then $\langle\Psi_0 \setminus \Psi\rangle = (h-1)(\alpha_0 - \nu)$, so $\alpha_0 - \nu$ is a dominant weight (by the argument given for $\langle\Psi\rangle$ at the beginning of this proof), and $\langle \alpha_0 - \nu, \check{\alpha}_0 \rangle = 0$ by the definition of Ψ_0. As mentioned earlier, this implies $\alpha_0 - \nu = 0$. Thus $\sigma = p\mu = -p\nu = -(h-1)\alpha_0 = -\langle \rho, \check{\alpha}_0 \rangle \alpha_0 = s_{\alpha_0} \cdot 0$. This completes the case $w = 1$.

The induction step is almost identical to that in Lemma 3.1.2(b). Write $w = s_\alpha w'$ as in that proof, and suppose as before that $w \cdot 0 + p\mu = -(\gamma_1 + \cdots + \gamma_m)$ for distinct $\gamma_1, \ldots, \gamma_m \in \Phi^+$. Then

$$w' \cdot 0 + p s_\alpha \mu = -(s_\alpha \gamma_1 + \cdots + s_\alpha \gamma_m + \alpha).$$

This is a sum of $m \pm 1$ distinct negative roots (according to whether or not some $\gamma_i = \alpha$). By induction, $w' \cdot 0 + p s_\alpha \mu = x' \cdot 0$ for some $x' \in W$. Apply $s_\alpha \cdot$ to get the result. □

3.6. In this section we prove results about complete reducibility of modules that will be later used in our cohomology calculations.

Proposition 3.6.1. *Let $p \geq h-1$, $w \in {}^J W$, and $\lambda \in \overline{C}_{\mathbb{Z}} \cap X^+$. Then*

(a) *$L_J(w \cdot 0)$ is in the bottom alcove for L_J;*
(b) *$L_J(w \cdot 0) \otimes L(\lambda)$ is completely reducible as an L_J-module.*

Proof. (a) First decompose $J := J_1 \cup J_2 \cup \cdots \cup J_t$ into indecomposable components, and let β_0 be the highest short root of one of the components $J_i =: K$. Observe that for $w \in {}^J W$,

$$\langle w \cdot 0 + \rho_K, \check{\beta}_0 \rangle = \langle w\rho - \rho + \rho_K, \check{\beta}_0 \rangle = \langle w\rho, \check{\beta}_0 \rangle = \langle \rho, w^{-1} \check{\beta}_0 \rangle$$

where in the second equality we have used that both ρ and ρ_K have inner product 1 with each simple coroot appearing in the decomposition of $\check{\beta}_0$. Now since $w \in {}^J W$ and $\beta_0 \in \Phi_J^+$, $w^{-1}\beta_0 \in \Phi^+$, and thus $0 \leq \langle \rho, w^{-1}\check{\beta}_0 \rangle \leq h-1 \leq p$. Hence, $w \cdot 0$ belongs to the closure of the bottom L_J alcove.

(b) Suppose that $L_J(\nu + \mu)$ is an L_J composition factor of $L_J(w \cdot 0) \otimes L(\lambda)$ where $\nu + \mu$ is J-dominant and ν is a weight of $L_J(w \cdot 0)$ and μ is a weight of $L(\lambda)$. We will show that $\nu + \mu$ belongs to the closure of the bottom L_J alcove. First observe that $\langle \mu, \check{\alpha} \rangle \leq \langle \lambda, \check{\alpha}_0 \rangle$ for all $\alpha \in \Phi$. Indeed, we can choose $w \in W$ such that $w\mu$ is dominant and since μ is a weight of $L(\lambda)$, $w\mu \leq \lambda$. Therefore,

$$\langle \mu, w^{-1}\check{\beta} \rangle = \langle w\mu, \check{\beta} \rangle \leq \langle w\mu, \check{\alpha}_0 \rangle \leq \langle \lambda, \check{\alpha}_0 \rangle$$

for all $\beta \in \Phi$.

Using the notation and results in (a), in addition to the fact that $\lambda \in \overline{C}_{\mathbb{Z}}$, we have

$$\begin{aligned}
\langle \nu + \mu + \rho_K, \check{\beta}_0 \rangle &= \langle \nu + \rho_K, \check{\beta}_0 \rangle + \langle \mu, \check{\beta}_0 \rangle \\
&\leq \langle w \cdot 0 + \rho_K, \check{\beta}_0 \rangle + \langle \mu, \check{\beta}_0 \rangle \\
&\leq (h-1) + \langle \lambda, \check{\alpha}_0 \rangle \\
&= \langle \rho, \check{\alpha}_0 \rangle + \langle \lambda, \check{\alpha}_0 \rangle \\
&= \langle \lambda + \rho, \check{\alpha}_0 \rangle \\
&\leq p.
\end{aligned}$$

The complete reducibility assertion follows by the Strong Linkage Principle [Jan, Proposition 6.13] because all the composition factors of $L_J(w \cdot 0) \otimes L(\lambda)$ are in the bottom L_J alcove. □

4. Kostant's Theorem and Generalizations

4.1. In this section we will prove Kostant's theorem, and its extension to characteristic p by Friedlander-Parshall ($p \geq h$) [FP1] and by Polo-Tilouine ($p \geq h-1$) [PT], for dominant highest weights in the closure of the bottom alcove. We begin by proving the result for trivial coefficients, and then use our tensor product results to prove it in the more general setting.

THEOREM 4.1.1. *Let $J \subseteq \Delta$. Assume $k = \mathbb{C}$ or $k = \overline{\mathbb{F}}_p$ with $p \geq h-1$. Then as an L_J-module*

$$\mathrm{H}^n(\mathfrak{u}_J, k) \cong \bigoplus_{\substack{w \in {}^J W \\ l(w)=n}} L_J(w \cdot 0).$$

PROOF. First observe that when $p = 2$ the condition that $p \geq h-1$ implies that $\Phi = A_1$ or A_2. For these cases the theorem can easily be verified directly. So assume that $p \geq 3$.

We first prove that every irreducible L_J-module in the sum on the right side is a composition factor of the left side. By the remarks at the beginning of Section 3.2, we have for each $w \in {}^J W$ with $l(w) = n$ the vector $f_{\Phi(w)} \in \mathrm{H}^n(\mathfrak{u}_J, k)$, where $\Phi(w) = \{\beta_1, \ldots, \beta_n\}$. To show that $f_{\Phi(w)}$ is a maximal vector for the Levi subalgebra $\mathfrak{l}_J$, fix $\gamma \in \Phi_J^+$. Then

$$x_\gamma f_{\Phi(w)} = \sum_{i=1}^{m} f_{\beta_1} \wedge \cdots \wedge x_\gamma f_{\beta_i} \wedge \cdots \wedge f_{\beta_m}. \tag{4.1.1}$$

Fix $\beta = \beta_i$ for some $1 \leq i \leq m$. For any root vector x_δ,

$$(x_\gamma f_\beta)(x_\delta) = -f_\beta([x_\gamma, x_\delta])$$

is nonzero if and only if $0 \neq [x_\gamma, x_\delta] \in \mathfrak{g}_\beta$, if and only if $\beta = \gamma + \delta$ (since root spaces are one-dimensional). Assume $x_\gamma f_\beta$ is nonzero; then it is a scalar multiple of f_δ where $\delta = \beta - \gamma$ is a root. Since $\beta \in \Phi(w)$, Proposition 3.2.1 implies that $\delta \in \Phi(w)$; that is, $\delta = \beta_j$ for some $j \neq i$. Thus $x_\gamma f_\beta = f_{\beta_j}$ already occurs in the wedge product in (4.1.1). So every term on the right hand side of (4.1.1) is 0, proving that $f_{\Phi(w)}$ is the highest weight vector of a L_J (resp. $(L_J)_1$) composition factor of $\mathrm{H}^n(\mathfrak{u}_J, k)$ when $k = \mathbb{C}$ (resp. $k = \overline{\mathbb{F}}_p$). But, the high weight is in the bottom L_J-alcove so we can conclude in general that this high weight corresponds to a L_J composition factor isomorphic to $L_J(w \cdot 0)$.

We now prove that all composition factors in cohomology appear in Kostant's formula. By Theorem 2.3.1 when $k = \mathbb{C}$, and by Theorem 2.4.1, Proposition 3.5.1, and Lemma 3.1.2 when $k = \overline{\mathbb{F}}_p$, any L_J composition factor of $\mathrm{H}^n(\mathfrak{u}_J, k)$ is an $L_J(w \cdot 0)$ for $w \in {}^J W$. By Lemmas 3.1.1(a) and 3.1.2(b), $l(w) = n$ and $L_J(w \cdot 0)$ occurs with multiplicity one in cohomology.

Moreover, when $k = \overline{\mathbb{F}}_p$, by Proposition 3.6.1 all the composition factors $L_J(w \cdot 0)$ lie in the bottom L_J alcove. By the Strong Linkage Principle, there are no nontrivial extensions between these irreducible L_J modules. So in either case, $\mathrm{H}^n(\mathfrak{u}_J, k)$ is completely reducible and given by Kostant's formula. □

We remark that the largest weight in $\Lambda^\bullet(\mathfrak{u}^*)$ is 2ρ. Moreover, $\langle 2\rho + \rho, \check{\alpha}_0 \rangle = 3\langle \rho, \check{\alpha}_0 \rangle = 3(h-1)$. This weight is not in the bottom alcove unless $p \geq 3(h-1)$. This necessitates a more delicate argument for the complete reducibility of the cohomology when $p \geq h-1$.

4.2. We can now use the previous theorem to compute the cohomology of $\mathfrak{u}_J$ with coefficients in a finite-dimensional simple $\mathfrak{g}$-module.

THEOREM 4.2.1. *Let $J \subseteq \Delta$ and $\mu \in X^+$. Assume that either $k = \mathbb{C}$, or $k = \overline{\mathbb{F}}_p$ with $\langle \mu + \rho, \check{\beta} \rangle \leq p$ for all $\beta \in \Phi^+$. Then as an L_J-module,*

$$\mathrm{H}^n(\mathfrak{u}_J, L(\mu)) \cong \bigoplus_{\substack{w \in {}^J W \\ l(w)=n}} L_J(w \cdot \mu).$$

PROOF. Observe that the conditions on μ imply $p \geq h - 1$. Namely, we have

$$p \geq \langle \mu + \rho, \check{\alpha}_0 \rangle = h - 1 + \langle \mu, \check{\alpha}_0 \rangle \geq h - 1. \tag{4.2.1}$$

For $p = 2$, the only case that remains to be checked is the case when $\Phi = A_1$ and $L(\mu) = L(1)$ is the two dimensional natural representation. This can be easily verified using the definition of cocycles and differentials in Lie algebra cohomology. So assume that $p \geq 3$.

First consider the case $k = \overline{\mathbb{F}}_p$ with $p = h - 1$. Then the inequalities in (4.2.1) must all be equalities, whence $\langle \mu, \check{\alpha}_0 \rangle = 0$. Since $\mu \in X^+$, it follows that $\mu = 0$. But now we are back to the setting of Theorem 4.1.1, where the result is proved. Thus for the rest of this proof we may assume $k = \mathbb{C}$ or $k = \overline{\mathbb{F}}_p$ with $p \geq h$.

We first prove that every L_J composition factor of the cohomology occurs in the direct sum on the right side. Let $L_J(\sigma)$ be an L_J composition factor of $\mathrm{H}^n(\mathfrak{u}_J, L(\mu))$. By Proposition 2.5.1 and Theorem 4.1.1, we have that $L_J(\sigma)$ is an L_J composition factor of $L_J(w \cdot 0) \otimes L(\mu)$ for some $w \in {}^J W$ with $l(w) = n$. Moreover, by definition $\mu \in X_1(T)$ and by the proof of Proposition 3.6.1(b), $\sigma \in (X_J)_1$. Hence, by Theorem 2.3.1 or 2.4.1, $\sigma = y \cdot \mu$ for some $y \in W_p$ (when $k = \mathbb{C}$ we set $W_p = W$).

According to Proposition 3.6.1, $L_J(w \cdot 0) = H^0_J(w \cdot 0)$, and $L_J(w \cdot 0) \otimes L(\mu)$ is completely reducible. Therefore, by using Frobenius reciprocity

$$0 \neq \mathrm{Hom}_{L_J}(L_J(\sigma), L_J(w \cdot 0) \otimes L(\mu)) \cong \mathrm{Hom}_{B_{L_J}}(L_J(\sigma), w \cdot 0 \otimes L(\mu)).$$

From this statement, one can see that

$$\sigma = y \cdot \mu = w \cdot 0 + \widetilde{\nu}$$

for some weight $\widetilde{\nu}$ of $L(\mu)$.

Choose $x \in W$ such that $\widetilde{\nu} = x\nu$ with ν dominant. Note that ν is still a weight of $L(\mu)$, so in particular $\nu \leq \mu$. Rewriting the previous equation gives

$$(w^{-1}y) \cdot \mu = w^{-1}x\nu.$$

Applying [Jan, Lemma II.7.7(a)] with $\lambda = 0$, $\nu_1 = \mu \in X(T)_+ \cap W(\mu - \lambda)$, we conclude that $\nu = \mu$. Now apply [Jan, Lemma II.7.7(b)] to conclude that there exists $w_1 \in W_p$ such that

$$w_1 \cdot 0 = 0 \quad \text{and} \quad w_1 \cdot \mu = w^{-1}x\mu.$$

But since $p \geq h$, ρ lies in the interior of the bottom alcove, so the stabilizer of 0 under the dot action of W_p is trivial; i.e., $w_1 = 1$. Thus $\mu = w^{-1}x\mu$, or equivalently, $w \cdot \mu = w \cdot 0 + x\mu = w \cdot 0 + \widetilde{\nu} = y \cdot \mu = \sigma$. Since $w \in {}^J W$ and $l(w) = n$, this proves that every composition factor in cohomology occurs in Kostant's formula (possibly with multiplicity greater than one).

We now prove that every L_J irreducible on the right side occurs as a composition factor in cohomology, with multiplicity one. Let $\sigma = w \cdot \mu$ for $w \in {}^JW$ with $l(w) = n$. The σ weight space of $C^\bullet = \Lambda^\bullet(\mathfrak{u}_J^*) \otimes L(\mu)$ contains at least the one dimensional space

$$\Lambda^n(\mathfrak{u}_J^*)_{w\cdot 0} \otimes L(\mu)_{w\mu}$$

since $w \cdot \mu = w \cdot 0 + w\mu = -\langle \Phi(w) \rangle + w\mu$. To see that this is the entire σ weight space of $C^\bullet$ we use a simple argument of Cartier [Cart], which we reproduce here for the reader's convenience.

Note first that there is a bijection between subsets $\Psi \subset \Phi^+$ and subsets $\widetilde{\Psi} \subset \Phi$ satisfying

$$\Phi = \widetilde{\Psi} \amalg -\widetilde{\Psi},$$

namely

$$\Psi = \widetilde{\Psi} \cap \Phi^+ \quad \text{and} \quad \widetilde{\Psi} = \Psi \cup -(\Phi^+ \smallsetminus \Psi).$$

Note that the collection of sets of the form $\widetilde{\Psi}$ is invariant under the ordinary action of W. It is easy to check that for such pairs,

$$\rho - \langle \Psi \rangle = -\tfrac{1}{2}\langle \widetilde{\Psi} \rangle. \tag{4.2.2}$$

Suppose $\sigma = -\langle \Psi \rangle + \nu$ for some $\Psi \subset \Phi^+$ and some weight ν of $L(\mu)$. It suffices to show $\Psi = \Phi(w)$ and $\nu = w\mu$. We have $\sigma + \rho = \rho - \langle \Psi \rangle + \nu = -\frac{1}{2}\langle \widetilde{\Psi} \rangle + \nu$.

Thus

$$\mu + \rho = w^{-1}(\sigma + \rho) = w^{-1}\nu - \tfrac{1}{2}\langle w^{-1}\widetilde{\Psi} \rangle = w^{-1}\nu - \langle \Gamma \rangle + \rho,$$

where we have applied (4.2.2) to $w^{-1}\widetilde{\Psi}$ and set $\Gamma = w^{-1}\widetilde{\Psi} \cap \Phi^+$.

But since $w^{-1}\nu$ is a weight of $L(\mu)$ we can write $w^{-1}\nu = \mu - \sum_i m_i\alpha_i$ with $m_i \in \mathbb{Z}_{\geq 0}$. So

$$\mu = \mu - \sum_i m_i \alpha_i - \langle \Gamma \rangle.$$

We conclude that all $m_i = 0$, so $w^{-1}\nu = \mu$ and $\nu = w\mu$. Also,

$$\Gamma = \varnothing \implies w^{-1}\widetilde{\Psi} = \Phi^- \implies \widetilde{\Psi} = w\Phi^- \implies \Psi = w\Phi^- \cap \Phi^+ = \Phi(w).$$

This is what we wanted to show.

Since the $w \cdot \mu$ weight space in the chain complex $C^\bullet$ is one dimensional and occurs in C^n, we conclude, as in the case of trivial coefficients, that $w \cdot \mu$ is a weight in the cohomology $\mathrm{H}^n(\mathfrak{u}_J, L(\mu))$. A corresponding weight vector in C^n is

$$v = f_{\Phi(w)} \otimes v_{w\mu},$$

where $f_{\Phi(w)}$ is as in the proof of Theorem 4.1.1 and $0 \neq v_{w\mu} \in L(\mu)_{w\mu}$. Fix $\gamma \in \Phi_J^+$; then $x_\gamma v = x_\gamma f_{\Phi(w)} \otimes v_{w\mu} + f_{\Phi(w)} \otimes x_\gamma v_{w\mu}$. We know from the proof of Theorem 4.1.1 that $x_\gamma f_{\Phi(w)} = 0$. Suppose $x_\gamma v_{w\mu}$ were not zero. Then it would be a weight vector in $L(\mu)$ of weight $w\mu + \gamma$. By W-invariance, $\mu + w^{-1}\gamma$ would be a weight of $L(\mu)$. But $w \in {}^JW$ and $\gamma \in \Phi_J^+$ imply $w^{-1}\gamma \in \Phi^+$, and this contradicts that μ is the highest weight of $L(\mu)$. Therefore v is annihilated by the nilradical of the Levi subalgebra, and hence its image in cohomology generates an L_J composition factor of $\mathrm{H}^n(\mathfrak{u}_J, L(\mu))$ isomorphic to $L_J(w \cdot \mu)$.

Note also that our argument proves that this composition factor occurs with multiplicity one.

The L_J highest weights are in the closure of the bottom L_J alcove by Propositions 2.5.1 and 3.6.1, and thus the cohomology is completely reducible as an L_J-module. $\square$

5. The Converse of Kostant's Theorem

5.1. Existence of extra cohomology. The following theorem shows that there are extra cohomology classes (beyond those given by Kostant's formula) that arise in $\mathrm{H}^\bullet(\mathfrak{u}, k)$ when char $k = p$ and $p < h - 1$. This can be viewed as a converse to Theorem 4.1.1 in the case when $J = \varnothing$. Examples in Section 6 will indicate that the situation is much more subtle for $J \neq \varnothing$ (i.e., extra cohomology classes may or may not arise depending on the size of J relative to the rank).

THEOREM 5.1.1. *Let* $k = \overline{\mathbb{F}}_p$ *with* $p < h - 1$. *Then* $\operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}, k) \neq \operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}, \mathbb{C})$.

PROOF. Fix a simple root α and let $J = \{\alpha\}$; shortly we will choose α more precisely. There exists a Lyndon-Hochschild-Serre spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(\mathfrak{u}/\mathfrak{u}_J, \mathrm{H}^j(\mathfrak{u}_J, k)) \Rightarrow \mathrm{H}^{i+j}(\mathfrak{u}, k).$$

Since $\dim \mathfrak{u}/\mathfrak{u}_J = 1$, $E_2^{i,j} = 0$ for $i \neq 0, 1$. Therefore, the spectral sequence collapses, yielding

$$\mathrm{H}^n(\mathfrak{u}, k) \cong \mathrm{H}^n(\mathfrak{u}_J, k)^{\mathfrak{u}/\mathfrak{u}_J} \oplus \mathrm{H}^1(\mathfrak{u}/\mathfrak{u}_J, \mathrm{H}^{n-1}(\mathfrak{u}_J, k)). \tag{5.1.1}$$

By the remarks at the beginning of Section 3.2, we can find explicit cocycles such that, as a T-module,

$$\bigoplus_{w \in W} w \cdot 0 \hookrightarrow \mathrm{H}^\bullet(\mathfrak{u}, k)$$

whereas by Lemmas 3.1.1 and 3.1.2, the only weights in $\mathrm{H}^\bullet(\mathfrak{u}_J, k)$ (or even in $\Lambda^\bullet(\mathfrak{u}_J^*)$) of the form $w \cdot 0$ with $w \in W$ occur when $w \in {}^J W$. So we must have

$$\bigoplus_{w \in W \smallsetminus {}^J W} w \cdot 0 \hookrightarrow \mathrm{H}^1(\mathfrak{u}/\mathfrak{u}_J, \mathrm{H}^\bullet(\mathfrak{u}_J, k)).$$

Thus it suffices to find "extra" cohomology in the first term on the right hand side of (5.1.1), meaning a cohomology class in characteristic p which does not have an analog in characteristic zero.

Since $\mathfrak{u}/\mathfrak{u}_J$ is isomorphic to the nilradical of the Levi subalgebra $\mathfrak{l}_J$, the first part of the proof of Theorem 4.1.1 shows that for $w \in {}^J W$ with $l(w) = n$, we have an explicit invariant vector of weight $w \cdot 0$ in $\mathrm{H}^n(\mathfrak{u}_J, k)^{\mathfrak{u}/\mathfrak{u}_J}$. Thus we get an inclusion

$$\bigoplus_{\substack{w \in {}^J W \\ l(w) = n}} w \cdot 0 \hookrightarrow \mathrm{H}^n(\mathfrak{u}_J, k)^{\mathfrak{u}/\mathfrak{u}_J} \subset \mathrm{H}^n(\mathfrak{u}_J, k).$$

By [Jan, Lemma 2.13] this induces an L_J-homomorphism from a sum of Weyl modules (for L_J)

$$\phi\colon S = \bigoplus_{\substack{w \in {}^J W \\ l(w) = n}} V_J(w \cdot 0) \to \mathrm{H}^n(\mathfrak{u}_J, k) \tag{5.1.2}$$

which is injective on the direct sum of the highest weight spaces. Next we claim that

$$\operatorname{Hd}_{L_J} \phi(S) = \bigoplus_{\substack{w\in {}^JW \\ l(w)=n}} L_J(w\cdot 0). \tag{5.1.3}$$

To see this, note first that $\phi(S) \cong S/\mathrm{Ker}\ \phi$, and $\mathrm{Ker}\ \phi \subset \operatorname{Rad}_{L_J} S$ because of the injectivity of ϕ on the highest weight spaces of the indecomposable direct summands $V_J(w\cdot 0)$ of S. This means that

$$\operatorname{Rad}_{L_J} \phi(S) \cong \operatorname{Rad}_{L_J}(S/\mathrm{Ker}\ \phi) = (\operatorname{Rad}_{L_J} S)/\mathrm{Ker}\ \phi.$$

Thus

$$\begin{aligned} \operatorname{Hd}_{L_J} \phi(S) &= \phi(S)/\operatorname{Rad}_{L_J} \phi(S) \\ &\cong (S/\mathrm{Ker}\ \phi)/((\operatorname{Rad}_{L_J} S)/\mathrm{Ker}\ \phi) \\ &\cong S/\operatorname{Rad}_{L_J} S \\ &\cong \bigoplus_{\substack{w\in {}^JW \\ l(w)=n}} L_J(w\cdot 0) \end{aligned}$$

as claimed.

Now choose α to be a short simple root, and fix $\widetilde{w} \in W$ such that $\widetilde{w}^{-1}\alpha = \alpha_0$, the highest short root. Then $\widetilde{w} \in {}^JW$ and

$$\langle \widetilde{w}\cdot 0+\rho, \check{\alpha}\rangle = \langle \widetilde{w}\rho, \check{\alpha}\rangle = \langle \rho, \widetilde{w}^{-1}\check{\alpha}\rangle = \langle \rho, \check{\alpha}_0\rangle = h-1 > p.$$

Thus $\lambda := \widetilde{w}\cdot 0$ is not in the restricted region for L_J. Write $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in (X_J)_1$ and $0 \neq \lambda_1 \in X_J^+$. There are two cases, according to whether or not $\phi(V_J(\lambda))$ is a simple L_J-module.

Case 1: $\phi(V_J(\lambda)) \cong L_J(\lambda)$. By Steinberg's tensor product theorem, $L_J(\lambda) \cong L_J(\lambda_0) \otimes L_J(\lambda_1)^{(1)}$. Since $\lambda_1 \neq 0$ (on J), $L_J(\lambda_1)^{(1)}$ has dimension at least two, and $\mathfrak{u}/\mathfrak{u}_J$ acts trivially on it. So this produces at least a two-dimensional space of vectors in $\mathrm{H}^n(\mathfrak{u}_J,k)^{\mathfrak{u}/\mathfrak{u}_J}$ arising from $L_J(\lambda)$ which produces "extra" cohomology.

Case 2: $N := \operatorname{Rad}_{L_J} \phi(V_J(\lambda)) \neq 0$. Then $N \subset \operatorname{Rad}_{L_J}\phi(S)$ and

$$0 \neq N^{\mathfrak{u}/\mathfrak{u}_J} \subset \phi(S)^{\mathfrak{u}/\mathfrak{u}_J} \subset \mathrm{H}^n(\mathfrak{u}_J,k)^{\mathfrak{u}/\mathfrak{u}_J}.$$

Since by (5.1.3) all the "characteristic zero" cohomology in $\mathrm{H}^n(\mathfrak{u}_J,k)^{\mathfrak{u}/\mathfrak{u}_J}$ has already been accounted for in $\operatorname{Hd}_{L_J}\phi(S)$, the vectors in $N^{\mathfrak{u}/\mathfrak{u}_J} \subset \operatorname{Rad}_{L_J}\phi(S)$ must be "extra" cohomology in characteristic p. □

5.2. Explicit extra cohomology. In this section we exhibit additional cohomology that arises in $\mathrm{H}^\bullet(\mathfrak{u},k)$ where $k=\overline{\mathbb{F}}_p$ in case $\Phi = A_n$.

THEOREM 5.2.1. *Let p be prime and Φ be of type A_n where $n=p+1$. Then the vector*

$$\sum_{i=1}^{p} f_{-\alpha_0}\wedge\gamma_1\wedge\gamma_2\wedge\cdots\wedge\widehat{\gamma_i}\wedge\cdots\wedge\gamma_p$$

appears as extra cohomology in $\mathrm{H}^{2p-1}(\mathfrak{u},k)$, where $\gamma_i = f_{-(\alpha_1+\cdots+\alpha_i)}\wedge f_{-(\alpha_{i+1}+\cdots+\alpha_n)}$.

Proof. Let $E = \sum_{i=1}^{p} f_{-\alpha_0} \wedge \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \widehat{\gamma}_i \wedge \cdots \wedge \gamma_p$. Consider the vector

$$f_{-\alpha_0} \wedge [d(f_{-\alpha_0})]^{p-1} := f_{-\alpha_0} \wedge \underbrace{d(f_{-\alpha_0}) \wedge d(f_{-\alpha_0}) \wedge \cdots \wedge d(f_{-\alpha_0})}_{p-1 \text{ times}},$$

with $d(f_{-\alpha_0}) = \gamma_1 + \gamma_2 + \cdots + \gamma_p \in \Lambda^2(\mathfrak{u}^*)$. First note by direct calculation one has $\gamma_i \wedge \gamma_j = \gamma_j \wedge \gamma_i$ and $\gamma_i \wedge \gamma_i = 0$. We can now apply the multinomial theorem for $[d(f_{-\alpha_0})]^m = [\gamma_1 + \gamma_2 + \cdots + \gamma_p]^m$ for $m \geq 2$:

$$\begin{aligned} [d(f_{-\alpha_0})]^m &= [\gamma_1 + \gamma_2 + \cdots + \gamma_p]^m \\ &= \sum_{r_1,\ldots,r_p} \binom{m}{r_1,\ldots,r_p} [\gamma_1]^{r_1} \wedge [\gamma_2]^{r_2} \wedge \cdots \wedge [\gamma_p]^{r_p} \end{aligned}$$

where $\sum_{i=1}^{p} r_i = m$ and $\binom{m}{r_1,\ldots,r_p} = \frac{m!}{r_1! r_2! \ldots r_p!}$. Consider the case when $m = p-1$. Since $[\gamma_i]^{r_i} = 0$ for $r_i \geq 2$, the only nonzero terms occur where $r_i = 0$ for some i, and $r_j = 1$ for all $j \neq i$. We have

$$\begin{aligned} [d(f_{-\alpha_0})]^{p-1} &= \sum_{i=1}^{p} \binom{p-1}{0,1,\ldots,1} (\gamma_1 \wedge \cdots \wedge \widehat{\gamma}_i \wedge \cdots \wedge \gamma_p) \\ &= (p-1)! \sum_{i=1}^{p} (\gamma_1 \wedge \cdots \wedge \widehat{\gamma}_i \wedge \cdots \wedge \gamma_p). \end{aligned}$$

So we have that $f_{-\alpha_0} \wedge [d(f_{-\alpha_0})]^{p-1} = (p-1)!\, E$. Since the terms in the above sum are linearly independent, this shows that $E \neq 0$. To prove that $E \in \operatorname{Ker} d$, we look at $f_{-\alpha_0} \wedge [d(f_{-\alpha_0})]^{p-1}$. Since d is a differential, $d(d(f_{-\alpha_0})) = 0$. Also note we can apply the multinomial theorem again to get $[d(f_{-\alpha_0})]^p = p!(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_i \wedge \cdots \wedge \gamma_p)$. Consequently,

$$\begin{aligned} d\left(f_{-\alpha_0} \wedge [d(f_{-\alpha_0})]^{p-1}\right) &= d(f_{-\alpha_0}) \wedge [d(f_{-\alpha_0})]^{p-1} - f_{-\alpha_0} \wedge d\left([d(f_{-\alpha_0})]^{p-1}\right) \\ &= [d(f_{-\alpha_0})]^p - f_{-\alpha_0} \wedge \left(\sum_{i=1}^{p-1} d(d(f_{-\alpha_0})) \wedge [d(f_{-\alpha_0})]^{p-2} \right) \\ &= p!\,(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_p). \end{aligned}$$

It now follows that

$$\begin{aligned} d(E) &= d\left(\frac{1}{(p-1)!} (f_{-\alpha_0} \wedge [d(f_{-\alpha_0})]^{p-1}) \right) \\ &= \frac{p!}{(p-1)!} (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_p) \\ &= p\,(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_p). \end{aligned}$$

Thus $d(E) = 0$ in characteristic p (but not in characteristic 0).

We need to verify that E is not in the image of the previous differential. This will follow by demonstrating that $\Lambda^{2p-2}(\mathfrak{u}^*)_{-p\alpha_0} = 0$ because the differentials respect weight spaces. Any weight in $\Lambda^{2p-2}(\mathfrak{u}^*)$ is of the form $\beta_1 + \beta_2 + \cdots + \beta_{2p-2}$ where the β_i are distinct negative roots. Observe that $\langle \beta_1 + \beta_2 + \cdots + \beta_{2p-2}, \check{\alpha}_0 \rangle \geq -2p+1$. One can deduce this because for each i, $\langle \beta_i, \check{\alpha}_0 \rangle = 0, \pm 1, \pm 2$ and is equal to -2 if and only if $\beta_i = -\alpha_0$. On the other hand, $\langle -p\alpha_0, \check{\alpha}_0 \rangle = -2p$, thus $\Lambda^{2p-2}(\mathfrak{u}^*)_{-p\alpha_0} = 0$. □

6. Examples for $\mathbf{H}^\bullet(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$

The following low rank examples were calculated using our computer package developed in MAGMA [BC, BCP]. Recall that the cohomology has a palindromic behavior so in the A_4 table the degrees are only listed up to half the dimension of $\mathfrak{u}_J$. Set $\mathrm{H}^n = \dim \mathrm{H}^n(\mathfrak{u}_J, k)$.

Type A_3, $h - 1 = 3$

J	p		H^0	H^1	H^2	H^3	H^4	H^5	H^6
$\varnothing$	0		1	3	5	6	5	3	1
	2		1	3	6	8	6	3	1
$\{1\}$ or $\{3\}$	0, 2		1	3	6	6	3	1	
$\{2\}$	0, 2		1	4	5	5	4	1	
$\{1,3\}$ or $\{2,4\}$	0, 2		1	4	6	4	1		
$\{1,2\}$ or $\{2,3\}$	0, 2		1	3	3	1			

Type A_4, $h - 1 = 4$

J	p		H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7
$\varnothing$	0		1	4	9	15	20	22	...	...
	2		1	4	11	25	38	42	...	...
	3		1	4	9	17	25	28	...	...
$\{1\}$ or $\{4\}$	0		1	4	10	19	26	...	...	...
	2		1	4	12	25	32	...	...	...
	3		1	4	10	20	27	...	...	...
$\{2\}$ or $\{3\}$	0		1	5	12	19	23	...	...	...
	2		1	5	12	23	33	...	...	...
	3		1	5	12	20	24	...	...	...
$\{1,3\}$ or $\{2,4\}$	0, 2, 3		1	6	13	23	30	...	...	...
$\{1,4\}$	0, 2, 3		1	4	14	25	28	...	...	...
$\{1,2\}$ or $\{3,4\}$	0, 3		1	4	12	18	...	...	...	...
	2		1	4	12	19	...	...	...	...
$\{2,3\}$	0, 3		1	6	14	14	...	...	...	...
	2		1	6	14	15	...	...	...	...
$\{1,3,4\}$ or $\{1,2,4\}$	0, 2, 3		1	6	15	20	...	...	...	...
$\{1,2,3\}$ or $\{2,3,4\}$	0, 2, 3		1	4	6	...	...	...	...	...

Type G_2, $h-1=5$

J	p		H^0	H^1	H^2	H^3	H^4	H^5	H^6
$\varnothing$	0		1	2	2	2	2	2	1
	2, 3		1	3	6	8	6	3	1
$\{1\}$	0, 2		1	4	5	5	4	1	
	3		1	4	7	7	4	1	
$\{2\}$	0		1	2	3	3	2	1	
	2		1	3	6	6	3	1	
	3		1	4	7	7	4	1	

7. Further Questions

The results in the preceding sections and our low rank examples naturally suggest the following open questions which are worthy of further study.

(7.1) Let G be a simple algebraic group over $\overline{\mathbb{F}}_p$ and $\mathfrak{g} = \operatorname{Lie} G$. Determine a maximal $c(J,p) > 0$ such that

$$\operatorname{ch} \mathrm{H}^n(\mathfrak{u}_J, \mathbb{C}) = \operatorname{ch} \mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$$

for $0 \leq n \leq c(J,p)$.

(7.2) Let $\Phi = A_n$ with $|\Delta| = n$.

a) Does $|\Delta - J| > p$ imply that $\operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \mathbb{C}) \neq \operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$?

b) Does $|\Delta - J| < p$ imply that $\operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \mathbb{C}) = \operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$?

We have seen that when $|\Delta - J| = p$ either conclusion can hold in the example where $\Phi = A_4$ and $|\Delta - J| = p = 2$.

c) What is the appropriate formulation of parts (a) and (b) when Φ is of arbitrary type?

(7.3) Let G be a simple algebraic group over $\overline{\mathbb{F}}_p$ and $\mathfrak{g} = \operatorname{Lie} G$. Assume that p is a good prime. Let $\mathcal{N}_1(\mathfrak{g}) = \{x \in \mathfrak{g} : x^{[p]} = 0\}$ (restricted nullcone). From work of Nakano, Parshall and Vella [NPV], there exists $J \subseteq \Delta$ such that $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ (i.e., closure of a Richardson orbit).

Does there exist $J \subseteq \Delta$ with $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ such that

$$\operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \mathbb{C}) = \operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}_J, \overline{\mathbb{F}}_p)?$$

(7.4) Let G be a simple algebraic group over $\overline{\mathbb{F}}_p$ and $\mathfrak{g} = \operatorname{Lie} G$. Compute

$$\operatorname{ch} \mathrm{H}^n(\mathfrak{u}_J, \overline{\mathbb{F}}_p)$$

for all p. It would be even better to describe the L_J-module structure.

Solving (7.4) would complete the analog of Kostant's theorem for the trivial module for all characteristics. One might be able to use (7.3) as a stepping stone to perform this computation. Moreover, this calculation would have major implications in determining cohomology for Frobenius kernels and algebraic groups (cf. [BNP]).

8. VIGRE Algebra Group at the University of Georgia

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