Relations among $\pi_1(X, x_0)$, $p^{-1}(x_0)$ and $G(\tilde{X})$

Let $X$ be a NICE space (i.e., $X$ is path-connected, locally path-connected, and semi-locally simply connected). Let

$$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$$

be a path-connected covering. (i.e., $\tilde{X}$ is path-connected). We want to understand relations between

$$\pi_1(X, x_0) \quad \text{the fundamental group}$$
$$G(\tilde{X}) \quad \text{the deck transformation group}$$
$$p^{-1}(x_0) \quad \text{the pre-image of the base-point}$$

We shall denote the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ by $H$.

$$\psi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$$

is given by $\alpha \longrightarrow \tilde{\alpha}(1)$.

Let $\alpha \in \pi_1(X, x_0)$. It is represented by a loop in $X$ based at $x_0$. Lift this loop to a path $\tilde{\alpha}$ in $\tilde{X}$ starting at $\tilde{x}_0$. In general, $\tilde{\alpha}$ is not a loop, but the end point $\tilde{\alpha}(1) \in p^{-1}(x_0)$. Thus, $\psi$ is defined by $\psi(\alpha) = \tilde{\alpha}(1)$.

• This $\psi$ is onto.

Let $\tilde{x}_1 \in p^{-1}(x_0)$. Find a path $\tilde{\gamma}$ in $\tilde{X}$ joining $\tilde{x}_0$ to $\tilde{x}_1$. Then $p \circ \tilde{\gamma}$ is a loop in $X$ based at $x_0$. Let $\alpha = [p \circ \tilde{\gamma}] \in \pi_1(X, x_0)$. Then, by construction, $\tilde{\psi}(\alpha) = \tilde{x}_1$.

• This $\psi$ is not one-one. Suppose $\alpha, \beta \in \pi_1(X, x_0)$ with $\psi(\alpha) = \psi(\beta)$. This means $\tilde{\alpha}(1) = \tilde{\beta}(1)$ so that $\tilde{\alpha} \tilde{\beta}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$. In other words,

$$\alpha \beta^{-1} = (p \circ \tilde{\alpha})(p \circ \tilde{\beta}) = p \circ (\tilde{\alpha} \ast \tilde{\beta}) \in p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H,$$

or equivalently, $\alpha H = \beta H$.

Conversely, suppose $\alpha H = \beta H$. Then $\alpha \beta^{-1} \in H$, and it lifts to a loop (not just a path) in $\tilde{X}$ based at $\tilde{x}_0$. That implies the end points of $\tilde{\alpha}$ and $\tilde{\beta}$ coincide. We have shown that $\tilde{\psi} : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow p^{-1}(x_0)$ factors through $\pi_1(X, x_0)/H$, and

$$\psi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_0)/H \overset{\tilde{\psi}}\longrightarrow p^{-1}(x_0) \quad \tilde{\psi} \text{ is bijective}$$
Now we try to relate $\pi_1(X,x_0)$ with the deck transformation group $G(\tilde{X})$. For $\alpha \in \pi_1(X,x_0)$, lift $\alpha$ to a path $\tilde{\alpha}$ in $\tilde{X}$ starting at $\tilde{x}_0$. Then $\tilde{\alpha}(1) \in p^{-1}(x_0)$.

Consider the lifting problem

\[
\begin{array}{ccc}
(\tilde{X}, \tilde{\alpha}(1)) & \xrightarrow{\phi(\alpha)} & (\tilde{X}, \tilde{x}_0) \\
p & & \downarrow p \\
(\tilde{X}, x_0) & \longrightarrow & (X, x_0)
\end{array}
\]

Such a $\phi(\alpha)$ exists if and only if $p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \subset p_* (\pi_1(\tilde{X}, \tilde{\alpha}(1)))$.

What is a relation between these two groups $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ and $p_* (\pi_1(\tilde{X}, \tilde{\alpha}(1)))$? Let $\tilde{\gamma}$ be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{\alpha}(1)$. Then $\gamma = p \circ \tilde{\gamma}$ is a loop in $X$, and

\[p_* (\pi_1(\tilde{X}, \tilde{\alpha}(1))) = \gamma^{-1} \cdot p_* (\pi_1(\tilde{X}, \tilde{x}_0)) \cdot \gamma.\]

Therefore, the condition becomes $H = \gamma^{-1} \cdot H \cdot \gamma$, or equivalently, $\gamma$ normalizes $H$. Thus, we have a map

\[\phi : N_{\pi_1(X,x_0)}(H) \longrightarrow G(\tilde{X})\]

$\phi$ is given by $\alpha \longrightarrow \phi(\alpha)$.

- This $\phi$ is onto.

Let $\tilde{f} : \tilde{X} \to X$ be a deck transformation, say with $\tilde{f}(\tilde{x}_0) = \tilde{x}_1$. Since $p = p \circ \tilde{f}$, we have a commuting diagram

\[
\begin{array}{ccc}
(\tilde{X}, \tilde{x}_1) & \xrightarrow{\tilde{f}} & (\tilde{X}, \tilde{x}_0) \\
p & & \downarrow p \\
(\tilde{X}, x_0) & \longrightarrow & (X, x_0)
\end{array}
\]

We wish to find $\alpha \in \pi_1(X,x_0)$ such that $\alpha \in N_{\pi_1(X,x_0)}(H)$ and $\phi(\alpha) = \tilde{f}$. Pick a path $\tilde{\alpha}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\alpha = p \circ \tilde{\alpha}$ is loop in $X$, and becomes an element of $\pi_1(X,x_0)$. From $p = p \circ \tilde{f}$, 

\[p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{\alpha}(1))).\]

Therefore, there is a lift $\phi(\alpha)$. Consequently, we have two maps $\tilde{f}$ and $\phi(\alpha)$ fitting the diagram. By the uniqueness of the lift, we see $\tilde{f} = \phi(\alpha)$.

- This $\phi$ is a homomorphism.

We have denoted a lift of $\alpha \in \pi_1(X,x_0)$ starting at $\tilde{x}_0$ simply by $\tilde{\alpha}$. Let’s write it by $\tilde{\alpha}_{\tilde{x}_0}$ (to denote the initial point). Then clearly,

\[\tilde{\alpha} \circ \beta_{\tilde{x}_0} = \tilde{\alpha}_{\tilde{x}_1} \circ \beta_{\tilde{x}_0}\]

where $\tilde{x}_1 = \beta_{\tilde{x}_0}(1)$. This proves that $\phi(\alpha \beta) = \phi(\alpha) \circ \phi(\beta)$. 
• This $\phi$ is not one-one.

Suppose $\alpha, \beta \in \pi_1(X, x_0)(H)$ and $\phi(\alpha) = \phi(\beta)$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Then $\tilde{\alpha} \ast \tilde{\beta}$ is a loop in $\tilde{X}$. Thus,

$$\alpha\beta^{-1} = p \circ (\tilde{\alpha} \ast \tilde{\beta}) \in p_\ast(\pi_1(\tilde{X}, \tilde{x}_0)) = H.$$ 

Conversely, if $\alpha\beta^{-1} \in H$, then the end points of the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ coincide. By the uniqueness of the lifts, we have $\phi(\alpha) = \phi(\beta)$. We proved:

\[
N_{\pi_1(X, x_0)}(H) \rightarrow N_{\pi_1(X, x_0)}(H)/H \xrightarrow{\tilde{\psi}} G(\tilde{X}) \quad \tilde{\psi} \text{ is bijective}
\]

**Group action**

**Definition** A group $G$ acts on a space $X$ if there is a map

$$\varphi : G \times X \rightarrow X$$

satisfying the following two conditions (writing $\varphi(g, x)$ by $g \cdot x$):

1. For every $x \in X$, $e \cdot x = x$ (where $e \in G$ is the identity element).
2. For every $g, h \in G$ and $x \in X$, $(gh) \cdot x = g \cdot (h \cdot x)$.

For a completely regular space, the set of all self homeomorphisms of $X$ becomes a group, denoted by $\text{TOP}(X)$. The above definition of an action is the same as having a group homomorphism

$$\psi : G \rightarrow \text{TOP}(X).$$

The condition (1) says $\psi(e)$ is the identity map of $X$; the condition (2) says $\psi$ is a group homomorphism.

An action is effective if, for every $g \in G$, there exists $x \in X$ such that $g \cdot x \neq x$. It is the same as the homomorphism $\psi$ is injective.

An orbit passing through $y \in X$ is the subset of $X$

$$Gy = \{g \cdot y : \ g \in G\} \subset X.$$ 

Note that $Gy \subset X$ and $G_y \subset G$.

The stabilizer (=isotropy subgroup) of the $G$-action at $y \in X$ is

$$G_y = \{g \in G : \ g \cdot y = y\} \subset G.$$
Suppose a group \( G \) acts on a space \( X \). The *orbit space* \( G \backslash X \) is the quotient \( X/\sim \) where
\[
x \sim y \text{ if and only if } \exists g \in G : y = g \cdot x
\]
(with the quotient topology).

**Example 1.** (a) Let \( \text{GL}_2 \mathbb{R} \) be the group of all non-singular \( 2 \times 2 \) matrices. Then it acts on \( \mathbb{R}^2 \) as matrix multiplications. Namely, for \( A \in \text{GL}_2 \mathbb{R} \) and \( x \in \mathbb{R}^2, \ A \cdot x = Ax \).

(b) The group \( G = \mathbb{R}^2 \) acts on the space \( X = \mathbb{R}^2 \) as
\[
a \cdot x = a + x
\]
for \( a \in G \) and \( x \in X \). So \( a \in \mathbb{R}^2 \) is a translation by \( a \). Inside \( G = \mathbb{R}^2 \), there is \( \mathbb{Z}^2 \) generated by \((1, 0)\) and \((0, 1)\). Then \( \mathbb{Z}^2 \) acts on \( \mathbb{R}^2 \). The orbit space is \( \mathbb{Z}^2 \backslash \mathbb{R}^2 \), a 2-torus.

(c) \( \text{Aff}(2) = \mathbb{R}^2 \rtimes \text{GL}_2 \mathbb{R} \) is defined, as a set \( \mathbb{R}^2 \times \text{GL}_2 \mathbb{R} \), but the group operation is given by
\[
(a, A) \cdot (b, B) = (a + Aa, AB).
\]
It is called the *affine group* of dimension 2. It acts on \( \mathbb{R}^2 \) as
\[
(a, A) \cdot x = a + Ax
\]
\( \text{Aff}(2) = \mathbb{R}^2 \rtimes \text{GL}_2 \mathbb{R} \) contains \( \text{E}(2) = \mathbb{R}^2 \rtimes \text{O}(2) \) (where \( \text{O}(2) \) is the *orthogonal group* \( (AA^T = I) \)). The group \( \text{E}(2) \) is called the *Euclidean isometry group*.

**Example 2.** Let \( \text{SL}_2 \mathbb{R} \) be the group of all \( 2 \times 2 \) matrices of determinant 1. It acts on \( \mathbb{R}^2 \) as matrix multiplications (since it is a subgroup of \( \text{GL}_2 \mathbb{R} \) in the above example).

There is a completely different action of \( \text{SL}_2 \mathbb{R} \) on \( \mathbb{H}^2 = \{ z = x + iy \in \mathbb{C} : y > 0 \} \), as linear fractional transformations. That is, for
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2 \mathbb{R} \text{ and } z \in \mathbb{H}
\]
\( A \cdot z \) is defined by
\[
A \cdot z = \frac{az + b}{cz + d}.
\]

**HW4.** Show this is an action. That is, show it satisfies the two conditions for the action.

It is not effective because \(-I \in \text{SL}_2 \mathbb{R}\) acts as the identity map. That is, the map \( \psi \) has a kernel \( \mathbb{Z}_2 = \{ \pm I \} \).

**Action of \( \pi_1(X, x_0) \) on \( p^{-1}(x_0) \)**

The action
\[
\pi_1(X, x_0) \times p^{-1}(x_0) \rightarrow p^{-1}(x_0)
\]
can be defined as follows: For $\alpha \in \pi_1(X, x_0)$ and $\tilde{x} \in p^{-1}(x_0)$,

$$\alpha \cdot \tilde{x} = \text{the end point of the lift of the loop } \alpha \text{ starting at } \tilde{x}$$

**HW1.** Prove this is an action.

**HW2.** This action is not effective. For example, if $\alpha \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, then $\alpha \cdot \tilde{x}_0 = \tilde{x}_0$. In fact, the isotropy subgroup of $\pi_1(\tilde{X}, \tilde{x}_0)$-action at the point $\tilde{x}_0$ is exactly $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

**HW3.** The “evaluation” of this action at the base-point $\tilde{x}_0$ is exactly the map $\psi : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$.

If $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a universal covering, then

$$\pi_1(X, x_0) \cong p^{-1}(x_0) \cong G(\tilde{X})$$

because $H$ is trivial.

Next: Properly discontinuous actions, Geometry