Sequences

Find a formula for each term in the following sequences:

- $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \dots\}$
- $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$
- $\{1, -1, 1, -1, 1, -1, \dots\}$
- $\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \dots\}$

Self Note: Talk about the Monotone Convergence Theorem.

Find the first four terms of the sequence and the limit:

- $a_n = \frac{3n}{1+6n}$
- $a_n = 1 + \frac{10^n}{9^n}$
- $a_n = \frac{3+5n^2}{n+n^2}$
- $a_n = n^2 e^{-n}$
- $a_n = n \sin(1/n)$
- $a_n = \frac{(2n-1)!}{(2n+1)!}$

Series

Find the sum of the following geometric series:

- $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$
- $\sum_{n=1}^{\infty} 3^{n+1} 4^{-n}$
- $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

Test the following series for divergence:

- $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$
- $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$
- $\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

Evaluate the following sum:

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$$

The Integral Test and the p-series

If f is a continuous, positive, decreasing function on $[1, \infty)$ and $a_n = f(n)$, then we know $\sum_{n=1}^{\infty} a_n$ is convergent exactly when $\int_1^{\infty} f(x)\, dx$ is convergent.

Test whether the following equations are convergent or not:

- $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$
- $\sum_{n=1}^{\infty} ke^{-k^2}$
- $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p-series (or the Riemann Zeta Function or Dirichlet Series) and is convergent if $p \geq 1$ but is divergent if $p < 1$.

Examine why this is true.

Comparison Tests

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Determine whether the following series are convergent or divergent.

- $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$
- $\sum_{n=0}^{\infty} \frac{9^{n+1}}{3+10^{n+1}}$
- $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

Pebbling a Chessboard (`https://www.youtube.com/watch?v=lFQGSGsXbXE`)

Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and positive, then both series diverge or converge.

Determine whether the following series are convergent or divergent.

- $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$
- $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Alternating Series

Suppose that $\sum(-1)^n a_n$ where a_n are all positive.
If the series is decreasing and the $\lim_{n\to\infty} a_n = 0$, then the series converges.

Determine whether the following series are convergent or divergent.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$
- $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$
- $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

Absolute Convergence

If $\sum |a_n|$ converges, we call $\sum a_n$ absolutely convergent.

If $\sum a_n$ converges but is not absolutely convergent, we call it conditionally convergent.

Fact: If a series is absolutely convergent, every rearrangement of the series is convergent to the same sum.

Ratio Test

- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is less than 1, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is greater than 1, $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is equal to 1, then we cannot conclude anything.

Notice! There are no assumptions! Our a_n don't need to be positive or decreasing or anything!

Determine whether the following series are convergent or divergent.

- $\sum_{n=1}^{\infty} \frac{n}{5^n}$
- $\sum_{k=1}^{\infty} ke^{-k}$
- $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ is less than 1, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ is greater than 1, $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ is equal to 1, then we cannot conclude anything.

Determine whether the following series are convergent or divergent.

- $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$
- $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$
- $\sum_{n=0}^{\infty} (\arctan n)^n$

Rearrangements

Two super neat facts:

- If a series is absolutely convergent, no matter what order you write the sum, you will get the same sum.
- If a series is convergent but not absolutely convergent, we can rearrange our series to get any number as the sum!

This second fact is called the Riemann Rearrangement Theorem and is John Urschel's favorite theorem. (A former football player for the Baltimore Ravens and current PhD candidate in Mathematics at MIT.)