Worksheet 4

① Find the Taylor series for $\sin x \cos x$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

So, $\sin x \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$

$\sin x$
$\cdot \cos x$ $\rightarrow$

$$\begin{array}{rcccc}
 & x & -\frac{x^3}{6} & +\frac{x^5}{120} & - \cdots \\
\cdot & 1 & -\frac{x^2}{2} & +\frac{x^4}{24} & -\frac{x^6}{720} \\
\hline
 & x & -\frac{x^3}{6} & +\frac{x^5}{120} & + \cdots \\
+ & & -\frac{x^3}{2} & +\frac{x^5}{12} & -\frac{x^7}{240} + \cdots \\
+ & & & \frac{x^5}{24} & -\frac{x^7}{144} + \cdots \\
+ & & & & -\frac{x^7}{720} + \cdots \\
\hline
 & x & -\frac{2x^3}{3} & +\frac{2x^5}{15} & -\frac{x^7}{80} + \cdots
\end{array}$$

I'm leaving space for the even powers but, in this case, I didn't need to.

So, $\sin x \cos x = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{x^7}{80} + \cdots$

② Find the Taylor series for $\dfrac{e^x}{\ln(1+x)}$. (Sorry, this was a bad problem)

$$\frac{e^x}{\ln(1+x)} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!}}{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}$$

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots \overline{\big)\, 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \dots}$$

This won't work since x doesn't divide 1.

New problem: $\dfrac{\ln(1+x)}{e^x}$ (This one is doable, so do this one instead)

$$
\begin{array}{r l}
 & x - \tfrac{3}{2}x^2 + \tfrac{4}{3}x^3 + \dots \\
\hline
1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \dots \big) & x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \dots \\
 & -(x + x^2 + \tfrac{x^3}{2} + \dots) \\
\hline
 & -\tfrac{3}{2}x^2 - \tfrac{1}{6}x^3 + \dots \\
 & -(-\tfrac{3}{2}x^2 - \tfrac{3}{2}x^3 + \dots) \\
\hline
 & \tfrac{4}{3}x^3
\end{array}
$$

$-\frac{1}{6} + \frac{9}{6} = \frac{8}{6} = \frac{4}{3}$

So $\boxed{\dfrac{\ln(1+x)}{e^x} = x - \tfrac{3}{2}x^2 + \tfrac{4}{3}x^3 + \dots}$

③ Compute the first 5 terms of the Taylor series for $\sqrt[3]{(2+x)^2}$.

$$\sqrt[3]{(2+x)^2} = (2+x)^{\frac{2}{3}} = 2^{\frac{2}{3}}\left(1+\frac{x}{2}\right)^{\frac{2}{3}}$$

$$= \boxed{2^{\frac{2}{3}} \sum_{n=0}^{\infty} \binom{\frac{2}{3}}{n} \left(\frac{x}{2}\right)^n}$$

$\boxed{n=0}$ $\quad 1$

$\boxed{n=1}$ $\quad \dfrac{\frac{2}{3}}{1!}\left(\dfrac{x}{2}\right) = \dfrac{1}{3}x$

$\boxed{n=2}$ $\quad \dfrac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)}{2!}\left(\dfrac{x}{2}\right)^2$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 x^2$$

$$= -\frac{1}{36}x^2$$

$\boxed{n=3}$ $\quad \dfrac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\dfrac{x}{2}\right)^3$

$$= \left(\frac{1}{6}\right)\left(\frac{\cancel{2}}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(\frac{1}{\cancel{8}}\right)x^3$$

$$= \frac{1}{72}x^3$$

$\boxed{n=4}$ $$\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\left(\frac{2}{3}-3\right)}{4!} \left(\frac{x}{2}\right)^4$$

$$= \frac{1}{24}\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(\frac{1}{16}\right)x^4$$

$$= -\frac{7}{3^5 \cdot 2^4} x^4$$

$$= -\frac{7}{3888} x^4$$

$$\sqrt[3]{(2+x)^2} \approx 2^{\frac{2}{3}}\left(1 + \frac{1}{3}x - \frac{1}{36}x^2 + \frac{1}{72}x^3 - \frac{7}{3888}x^4 \ldots\right)$$

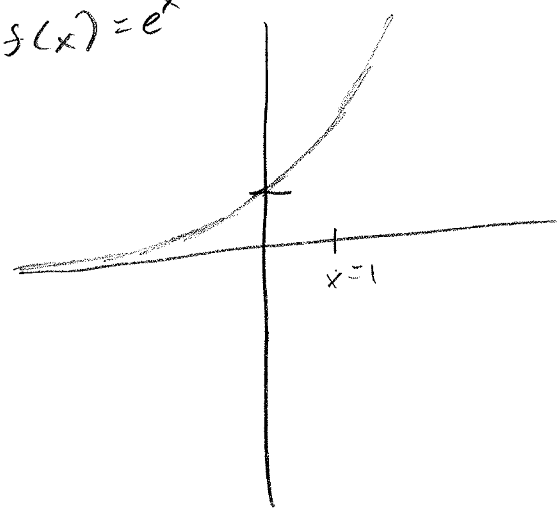
Remainders

① Find a bound on the remainder if you compute the Taylor series $f(x)=e^x$ out to 5 terms at $x=0$ and $x=1$.

$\boxed{x=0}$ $c_0 = f(0)$, so our Taylor series is always exact at the center!

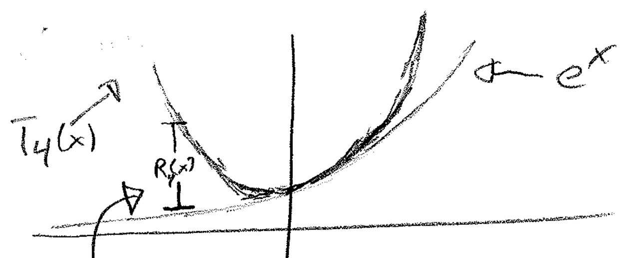
$\boxed{x=1}$

$f(x)=e^x$



$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So, $T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$



$R_4(x)$, the remainder is the difference between $T_4(x)$ and e^x, i.e. the error.

$$R_4(x) = e^x - T_4(x).$$

We will not always be able to find $R_4(x)$, but we can find a bound on $R_4(x)$ (in otherwords a number $R_4(x)$ has to be less than!

To do this, we have to fix an interval around our center. Since we are looking to estimate $f(1)$
So we need to have an interval including $x=1$.

So, we pick $[-1, 1]$.

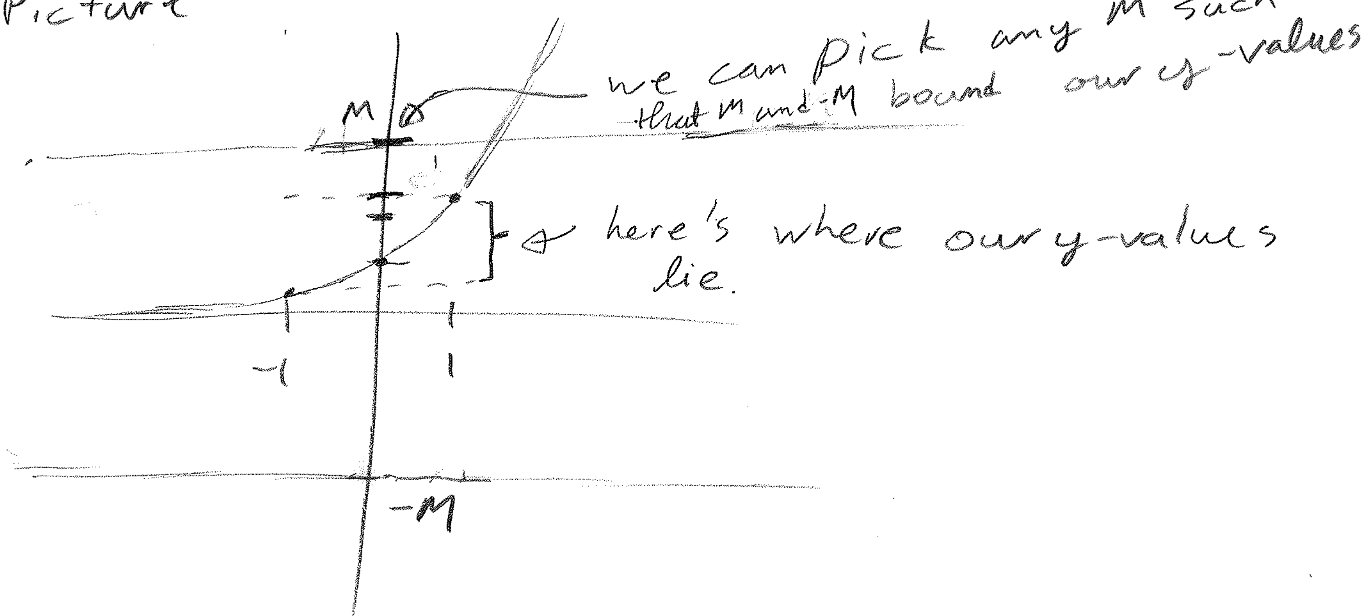
We could also have picked $[-2, 2]$ or $[-3000, 3000]$ if we *want*, but the smaller the interval, the easier.

Next, we find the 5th derivative:

$$f^{(5)}(x) = e^x$$

and we find a bound for e^x, M.

Picture



Here, we can pick $M = e^1$

Using the formula, we get

$$|R_4(x)| \leq \frac{e^1}{5!}|x|^5 \quad \text{for } -1 \leq x \leq 1.$$

So, at $x=1$:

$$\boxed{|R_4(1)| \leq \frac{e^1}{5!}|1|^5 = \frac{e}{5!} \approx .0227}$$

So our error is no more than .0227.

If we want a better estimate, we can compute more terms.

(2) Find a bound on the remainder if you compute the Taylor series $f(x) = \sin x$ out to 12 terms at any x.

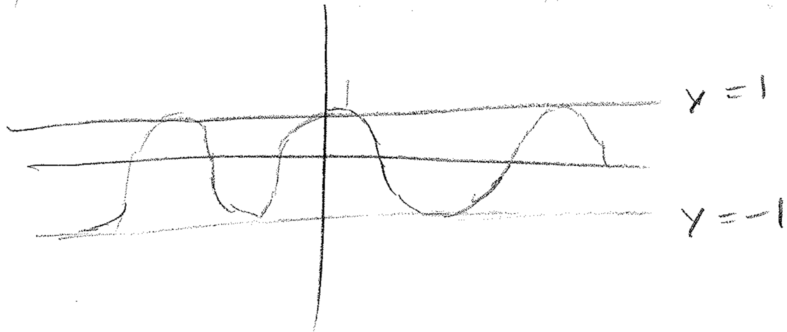
$$|R_{12}(x)| \leq \frac{M}{13!}|x|^{13} \quad \text{for} \quad -d \leq x \leq d$$

Let's pick a number, $x = b$.

Then our interval is $-b \leq x \leq b$ (This isn't too important since the entire function will be bounded by $M=1$)

$f^{(13)}(x) = \cos x$, so

$-1 \leq f^{(13)}(x) \leq 1$ and so $M = 1$



Note: we didn't have to compute $f^{(13)}(x)$ since it will be $\sin x$, $\cos x$, $-\sin x$, or $-\cos x$ which are all bounded by 1.

Thus, $|R_{12}(b)| \leq \frac{1}{12!} b^{12}$