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Chapter 1

Introduction

Complex semisimple Lie algebras have the complete reducibility property. Each complex finite-dimensional irreducible representation of a complex semisimple Lie algebra is parameterized by a highest weight. Each finite-dimensional irreducible representation also has a unique weight diagram including a specific multiplicity for each weight; the multiplicity of a weight is equal to the dimension of the corresponding weight space in the representation. These multiplicities have been the topic of many research efforts. Several formulas for computing these multiplicities have been developed by Freudenthal [4], Kostant [9], Lusztig [12], Littelmann [11], and Sahi [14]. Many of these formulas are general and recursive.

In Chapter 2, we will focus on the weight multiplicities of finite-dimensional representations of the classical rank two Lie algebra $\mathfrak{sp}(4, \mathbb{C})$ corresponding to the Lie group Sp(4). These multiplicities are surprisingly difficult to obtain considering there is a nice formula for the weight multiplicities for another classical rank two Lie algebra, $\mathfrak{sl}(3, \mathbb{C})$. In 2004 in [3], Cagliero and Tirao gave an explicit closed formula for the weight multiplicities of any irreducible representation of this Lie algebra, and to the best of our knowledge, this was the first paper to do so. The method of proof in [3] employed a Howe duality theorem and the explicit decomposition of tensor products of exterior powers of fundamental representations of Sp(4). In this note, we will provide an alternate, elementary approach to finding an explicit closed formula for the weight multiplicities of any irreducible representation of $\mathfrak{sp}(4, \mathbb{C})$.

We first present a useful identity between finite-dimensional representations of the rank 2 symplectic Lie algebra. In Section 2.2, using a basic approach, we develop this first identity. It is based on a general result involving multilinear algebra for symmetric tensors; see Proposition 2.1 and Corollary 2.2 from Section 2.1. While these are certainly well known to experts, we have included proofs for completeness. Proposition 2.3 (and subsequently Corollary 2.4) follows from this together with the explicit determination of certain highest weight vectors occurring in a tensor product of symmetric powers of the standard representation of $\mathfrak{sp}(4, \mathbb{C})$. Corollary 2.4 then shows how an irreducible representation can be expressed as a linear combination of tensor products of symmetric powers of the standard representation. These results can also be found using Littelmann's paper [10] and Young tableaux or using a formula involving characters from Section 24.2 in [5].

In Section 2.3, we determine the weight multiplicities of any dominant weight in a tensor product of symmetric powers of the standard representation. In Section 2.4, we use the results of Sections 2.2 and 2.3 to create an explicit closed formula for the weight multiplicities of the dominant weights in any irreducible representation of $\mathfrak{sp}(4, \mathbb{C})$.

In Chapter 3, Section 3.1, we introduce the concepts of L- and ε-factors for Sp(4), which are calculated for a given representation of Sp(4) and a fixed representation of the real Weil group, $\zeta : W_{\mathbb{R}} \to \text{Sp}(4)$, parameterized by two odd integers k and l. As usual, we only need to consider irreducible representations of Sp(4). The results of Section 2.2 can be adapted to reduce the problem of calculating the archimedean factors of an irreducible representation to the determination of the archimedean factors of a tensor product of symmetric powers of the standard representation. The L- and ε-factors of a representation require explicit multiplicity information. Theorem 2.5 is then recalled to help with this calculation of archimedean factors of a tensor product of symmetric powers of the standard representation. Section 3.2 contains a description of how to calculate the L- and ε-factors of any representation of Sp(4).

Any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation. This is the main result of Chapter 4 along with an algorithm for determining such formal sums and two formulas. These results are already known as in [5], Section 24.2, by appropriately interpreting a proposition involving the character of an irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$, but we will provide an alternate approach using Littelmann's paper [10] and combinatorial arguments.

In [1] and [16], the authors, Akin and Zelevinskii respectively, independently prove an identity expressing any irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ as a formal sum of tensor products of symmetric powers of the standard representation using resolutions, so the ability to write an irreducible representation as a formal sum of tensor products of symmetric powers of the standard representation has been of interest for other classical Lie algebras as well.

In Section 4.1 we present a useful identity between finite-dimensional representations of the rank m symplectic Lie algebra by generalizing the results of Section 2.2. Proposition 4.1 (and subsequently Corollary 4.2) follows from the general multilinear algebra results of Section 2.1 together with the explicit determination of certain highest weight vectors occurring in a tensor product of symmetric powers of the standard representation of $\mathfrak{sp}(2m, \mathbb{C})$. Corollary 4.2 then shows how an irreducible representation with particular highest weights can be expressed as a linear combination of tensor products of symmetric powers of the standard representation.

In Section 4.2, Littelmann's generalization of the Littlewood-Richardson rule in [10] is applied to $\mathrm{Sp}(2m)$ to prove the main result of Chapter 4, Theorem 2.5. This theorem states that any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation, and the method of proof creates an algorithm for finding such a sum. In Section 4.3, we present a refinement of the algorithm from the proof along with a formula, which simplifies the process for finding the formal sum.

In Section 4.4, we show examples for the symplectic Lie algebras of rank 2 and 3 using the results of Sections 4.2 and 4.3. At the end of Chapter 4 in Section 4.5, we present a final formula that explicitly determines the formal sum for a general case.

Chapter 2

A closed formula for weight multiplicities for $\mathfrak{sp}(4, \mathbb{C})$

2.1 A result on symmetric tensors

For a positive integer n, let S_n be the symmetric group on n letters. For this section, let V be a finite-dimensional vector space over a field with characteristic zero, F. S_n acts linearly on $V^{\otimes n}$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$. Let $\mathrm{sym} : V^{\otimes n} \to V^{\otimes n}$ be the usual symmetrization map, i.e., $\mathrm{sym}(v) = \sum_{\sigma \in S_n} \sigma(v)$. The kernel of this map is spanned by all elements of the form $v - \sigma(v)$ for $v \in V^{\otimes n}$ and $\sigma \in S_n$. We denote by $\mathrm{Sym}^n(V)$ the image of sym or equivalently the quotient of $V^{\otimes n}$ by the kernel of sym.

The proof of this definition of the kernel is as follows.

$$\begin{aligned}\mathrm{sym}(v - \sigma(v)) &= \sum_{\tau \in S_n} \tau(v - \sigma(v)) = \sum_{\tau \in S_n} \tau(v) - \sum_{\tau \in S_n} \tau\sigma(v) \\ &= \sum_{\tau \in S_n} \tau(v) - \sum_{\tau \in S_n} (\tau\sigma^{-1})\sigma(v) = \sum_{\tau \in S_n} \tau(v) - \sum_{\tau \in S_n} \tau(v) = 0.\end{aligned}$$

Therefore, $\ker(\text{sym}) \supset \langle v - \sigma(v) \mid v \in V^{\otimes n}, \sigma \in S_n \rangle$.

If $v \in \ker(\text{sym})$, then $\sum_{\sigma \in S_n} \sigma(v) = 0$. So $v + \sum_{\sigma \in S_n, \sigma \neq id} \sigma(v) = 0$, and

$$v = -\sum_{\sigma \in S_n, \sigma \neq id} \sigma(v).$$

Then

$$n!v = (n! - 1)v + v = (n! - 1)v - \sum_{\sigma \in S_n, \sigma \neq id} \sigma(v) = \sum_{\sigma \in S_n, \sigma \neq id} (v - \sigma(v)),$$

and $v = \frac{1}{n!} \sum_{\sigma \in S_n, \sigma \neq id} (v - \sigma(v))$. Therefore, $\ker(\text{sym}) \subset \langle v - \sigma(v) \mid v \in V^{\otimes n}, \sigma \in S_n \rangle$, and $\ker(\text{sym}) = \langle v - \sigma(v) \mid v \in V^{\otimes n}, \sigma \in S_n \rangle$. Hence,

$$\text{Sym}^n V \cong V^{\otimes n} / \langle v - \sigma(v) \mid v \in V^{\otimes n}, \sigma \in S_n \rangle.$$

$\text{Sym}^{m_1} V_1 \otimes \ldots \otimes \text{Sym}^{m_n} V_n$ is the tensor product of $\text{Sym}^{m_i} V_i, 1 \leq i \leq n$, as previously defined as the image of the symmetrization map. This is equivalent to defining $\text{Sym}^{m_1} V_1 \otimes \ldots \otimes \text{Sym}^{m_n} V_n$ to be the image of the map

$$\text{sym} \otimes \ldots \otimes \text{sym} : V_1^{\otimes m_1} \otimes \ldots \otimes V_n^{\otimes m_n} \to V_1^{\otimes m_1} \otimes \ldots \otimes V_n^{\otimes m_n}$$

such that $v_1 \otimes \ldots \otimes v_n \mapsto \sum_{\sigma_1 \in S_{m_1}} \sigma_1(v_1) \otimes \ldots \otimes \sum_{\sigma_n \in S_{m_n}} \sigma_n(v_n)$ where $\sigma_i(v_i)$ is defined linearly by $\sigma_i(\alpha_1 \otimes \ldots \otimes \alpha_{m_i}) = \alpha_{\sigma_i^{-1}(1)} \otimes \ldots \otimes \alpha_{\sigma_i^{-1}(n)}$. This map is well-defined by the universal property for tensor products since this map is linear in each of the components $V_i^{\otimes m_i}$ by the linearity of the sym map. Then $\text{Sym}^{m_1} V_1 \otimes \ldots \otimes \text{Sym}^{m_n} V_n$ is isomorphic to $V_1^{\otimes m_1} \otimes \ldots \otimes V_n^{\otimes m_n} / \ker(\text{sym} \otimes \ldots \otimes \text{sym})$. The kernel of $\text{sym} \otimes \ldots \otimes \text{sym}$ is equal to $\sum_{i=1}^{n} \langle v_1 \otimes \ldots \otimes (v_i - \sigma_i(v_i)) \otimes \ldots \otimes v_n \mid v_i \in V_i^{\otimes m_i}, \sigma_i \in S_{m_i} \rangle$.

Also, since $(A_1/B_1 \otimes \ldots \otimes A_n/B_n) \cong (A_1 \otimes \ldots \otimes A_n)/(\sum_{i=1}^{n} A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \otimes A_{i+1} \otimes \ldots \otimes A_n)$ using the isomorphism $[a_1] \otimes \ldots \otimes [a_n] \mapsto [a_1 \otimes \ldots \otimes a_n]$, $\mathrm{Sym}^{m_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n} V_n \cong (V_1^{\otimes m_1}/\langle v_1 - \sigma_1(v_1) \mid v_1 \in V_1^{\otimes m_1}, \sigma_1 \in S_{m_1}\rangle) \otimes \ldots \otimes (V_n^{\otimes m_n}/\langle v_n - \sigma_n(v_n) \mid v_n \in V_n^{\otimes m_n}, \sigma_n \in S_{m_n}\rangle) \cong (V_1^{\otimes m_1} \otimes \ldots \otimes V_n^{\otimes m_n})/(\sum_{i=1}^{n} \langle v_1 \otimes \ldots \otimes (v_i - \sigma_i(v_i)) \otimes \ldots \otimes v_n \mid v_i \in V_i^{\otimes m_i}, \sigma_i \in S_{m_i}\rangle)$.

Hence, $\mathrm{Sym}^{m_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n} V_n \cong (V_1^{\otimes m_1} \otimes \ldots \otimes V_n^{\otimes m_n})/(\sum_{i=1}^{n} \langle v_1 \otimes \ldots \otimes (v_i - \sigma_i(v_i)) \otimes \ldots \otimes v_n \mid v_i \in V_i^{\otimes m_i}, \sigma_i \in S_{m_i}\rangle)$.

Proposition 2.1. *Let $V_1, \ldots, V_n$ be finite-dimensional representations of a Lie algebra L. For some one-dimensional subspace U of $V_1 \otimes \ldots \otimes V_n$ generated by the element α, for any $m_i \geq 1$ define*

$$\phi : \mathrm{Sym}^{m_1 - 1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n - 1} V_n \to \mathrm{Sym}^{m_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n} V_n$$

as multiplication by the element α. Then ϕ is an injective, intertwining map.

Proof. Let V_i be a finite-dimensional representation with basis $\{v_1^{(i)}, v_2^{(i)}, \ldots, v_{k_i}^{(i)}\}$. Then U is the one-dimensional subspace of $V_1 \otimes \ldots \otimes V_n$ generated by

$$\sum_t \gamma_t^{(1)} \otimes \ldots \otimes \gamma_t^{(n)} = \sum_{(j_1 \times \ldots \times j_n)} a(j_1 \times \ldots \times j_n) v_{j_1}^{(1)} \otimes \ldots \otimes v_{j_n}^{(n)}$$

for some constants $a(j_1 \times \ldots \times j_n)$ where $(j_1 \times \ldots \times j_n)$ runs over the set

$$\{1, \ldots, k_1\} \times \ldots \times \{1, \ldots, k_n\}.$$

Now, ϕ is the linear map such that

$$\begin{aligned}
&\mathrm{sym}(\alpha_1^{(1)} \otimes \ldots \otimes \alpha_{m_1-1}^{(1)}) \otimes \ldots \otimes \mathrm{sym}(\alpha_1^{(n)} \otimes \ldots \otimes \alpha_{m_n-1}^{(n)}) \\
&\quad \mapsto \sum_{(j_1\times\ldots\times j_n)} a(j_1 \times \ldots \times j_n)\mathrm{sym}(\alpha_1^{(1)} \otimes \ldots \otimes \alpha_{m_1-1}^{(1)} \otimes v_{j_1}^{(1)}) \otimes \ldots \\
&\qquad\qquad \otimes \mathrm{sym}(\alpha_1^{(n)} \otimes \ldots \otimes \alpha_{m_n-1}^{(n)} \otimes v_{j_n}^{(n)}).
\end{aligned}$$

Since $U \neq 0$, some $a(j_1 \times \ldots \times j_n) \neq 0$. Without loss of generality, assume $a_{(1\times\ldots\times 1)} \neq 0$. The linear map ϕ is well-defined because a permutation of the $\alpha_i^{(j)}$ vectors in $\mathrm{sym}(\alpha_1^{(j)} \otimes \ldots \otimes \alpha_{n-1}^{(j)})$ yields the same element and equivalently the same permutation of the $\alpha_i^{(j)}$ vectors in $\mathrm{sym}(\alpha_1^{(j)} \otimes \ldots \otimes \alpha_{n-1}^{(j)} \otimes \gamma_t^{(j)})$ yields the same element.

We will now show directly that ϕ is injective. For the given bases of V_i, identify the standard basis elements of $\mathrm{Sym}^{m_1-1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n-1}V_n$ as $k_1 \times \ldots \times k_n$ - tuples $(c_1^{(1)}, \ldots, c_{k_1}^{(1)}) \times \ldots \times (c_1^{(n)}, \ldots, c_{k_n}^{(n)})$ or $\Pi_{i=1}^n(c_1^{(i)}, \ldots, c_{k_i}^{(i)})$ such that for a particular basis element $c_i^{(j)}$ is equal to the number of times $v_i^{(j)}$ appears in that basis element. The standard basis for $\mathrm{Sym}^{m_1-1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n-1}V_n$ is equivalent to the set $\{\Pi_{i=1}^n(c_1^{(i)}, \ldots, c_{k_i}^{(i)}) \mid c_i^{(j)} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k_j} c_i^{(j)} = m_j - 1, 1 \leq j \leq n\}$. Similarly, identify the standard basis elements of $\mathrm{Sym}^{m_1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n}V_n$ as $k_1 \times \ldots \times k_n$ - tuples $\Pi_{i=1}^n(d_1^{(i)}, \ldots, d_{k_i}^{(i)})$ such that for a particular basis element $d_i^{(j)}$ is equal to the number of times $v_i^{(j)}$ appears in that basis element. The standard basis for $\mathrm{Sym}^{m_1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n}V_n$ is equivalent to the set $\{\Pi_{i=1}^n(d_1^{(i)}, \ldots, d_{k_i}^{(i)}) \mid d_i^{(j)} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k_j} d_i^{(j)} = m_j, 1 \leq j \leq n\}$. Therefore any element, $\mathbf{v}$, of $\mathrm{Sym}^{m_1-1}V_1 \otimes \ldots \otimes$

$\text{Sym}^{m_n-1}V_n$ has the form

$$\mathbf{v} = \sum_{\Pi_{i=1}^n(c_1^{(i)},\ldots,c_{k_i}^{(i)})} b(\Pi_{i=1}^n(c_1^{(i)},\ldots,c_{k_i}^{(i)}))(\Pi_{i=1}^n(c_1^{(i)},\ldots,c_{k_i}^{(i)}))$$

for some constants $b(\Pi_{i=1}^n(c_1^{(i)},\ldots,c_{k_i}^{(i)}))$. Let $\mathbf{v}$ be an element of the kernel of ϕ with this form. Then $\phi(\mathbf{v}) = 0$, and we will now show that every

$$b(\Pi_{i=1}^n(c_1^{(i)},\ldots,c_{k_i}^{(i)})) = 0.$$

Thus proving $\ker\phi = \{0\}$.

Since $\Pi_{i=1}^n(d_1^{(i)},\ldots,d_{k_i}^{(i)})$ is a basis element of $\text{Sym}^{m_1}V_1 \otimes \ldots \otimes \text{Sym}^{m_n}V_n$, its coefficent in $\phi(\mathbf{v})$ is

$$\sum_{d_{j_i}^{(i)}\neq 0} a(j_1 \times \ldots \times j_n) b(\Pi_{i=1}^n(d_1^{(i)},\ldots,d_{j_i}^{(i)}-1,\ldots,d_{k_i}^{(i)})) = 0.$$

Let $S = r_1 + \ldots + r_n$. We will now prove by induction on S that

$$b(\Pi_{i=1}^n(m_i - r_i, c_2^{(i)},\ldots,c_{k_i}^{(i)})) = 0,$$

$1 \leq r_i \leq m_i$. This covers all basis elements of $\text{Sym}^{m_1-1}V_1 \otimes \ldots \otimes \text{Sym}^{m_n-1}V_n$ since every entry in the i-th component of a basis element is between 0 and $m_i - 1$.

Consider the base case where $r_i = 1$ for all i and then $S = n$. The only basis element with $r_i = 1$ for all i is $\Pi_{i=1}^n(m_i - 1, 0, \ldots, 0)$. The coefficient of the basis element $\Pi_{i=1}^n(m_i, 0, \ldots, 0)$ in $\text{Sym}^{m_1}V_1 \otimes \ldots \otimes \text{Sym}^{m_n}V_n$ for $\phi(\mathbf{v})$ is

$$a(1 \times \ldots \times 1) b(\Pi_{i=1}^n(m_i - 1, 0, \ldots, 0)) = 0.$$

Since $a(1 \times \ldots \times 1) \neq 0$, $b(\Pi_{i=1}^{n}(m_i - 1, 0, \ldots, 0)) = 0$.

Assume $b(\Pi_{i=1}^{n}(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})) = 0$ for all $S = r_1 + \ldots + r_n \leq s$. Now let $S = s+1$. Consider any particular n-tuple $(r_1, \ldots, r_n)$ such that $S = \sum_{i=1}^{n} r_i = s+1$. It is enough to show $b(\Pi_{i=1}^{n}(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})) = 0$ for any basis vector in $\mathrm{Sym}^{m_1-1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n-1}V_n$ of the form $\Pi_{i=1}^{n}(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})$. Consider the coefficient of $\Pi_{i=1}^{n}(m_i - r_i + 1, c_2^{(i)}, \ldots, c_{k_i}^{(i)})$ in $\mathbf{v}$, which is equal to 0.

$$
\begin{aligned}
&a(1 \times \ldots \times 1) b(\Pi(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})) \\
&\quad + \sum_{c_{j_i}^{(i)} \neq 0, \text{ not all } j_i = 1} a(j_1 \times \ldots \times j_n) b(\Pi(m_i - r_i + 1, c_2^{(i)}, \ldots, c_{j_i}^{(i)} - 1, \ldots, c_{k_i}^{(i)})) \\
&= 0.
\end{aligned}
$$

Inside the sum over $c_{j_i}^{(i)} \neq 0$, not all $j_i = 1$, consider a particular coefficient $b(\Pi(m_i - (r_i - 1), c_2^{(i)}, \ldots, c_{j_i}^{(i)} - 1, \ldots, c_{k_i}^{(i)}))$. Since not all j_i are equal to 1, there is some t such that $j_t \neq 1$. This means that for this term, $S \leq (\sum_{i \neq t} r_i) + r_t - 1 = (\sum_{i=1}^{n} r_i) - 1 = (s+1) - 1 = s$, which satisfies the induction hypothesis. Therefore, $b(\Pi(m_i - r_i + 1, c_2^{(i)}, \ldots, c_{j_i}^{(i)} - 1, \ldots, c_{k_i}^{(i)})) = 0$ when not all j_i are equal to 1.

Therefore

$$
\sum_{c_{j_i}^{(i)} \neq 0, \text{ not all } j_i = 1} a(j_1 \times \ldots \times j_n) b(\Pi(m_i - r_i + 1, c_2^{(i)}, \ldots, c_{j_i}^{(i)} - 1, \ldots, c_{k_i}^{(i)})) = 0,
$$

and the only term left in the previous sum is

$$
a(1 \times \ldots \times 1) b(\Pi(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})) = 0.
$$

Since $a(1 \times \ldots \times 1) \neq 0$, $b(\Pi(m_i - r_i, c_2^{(i)}, \ldots, c_{k_i}^{(i)})) = 0$, which proves the inductive step.

To prove injectivity an alternate way, let $\mathrm{Sym}(V)$ be the algebra $\bigoplus_{x=0}^{\infty} \mathrm{Sym}^x V$. Then $\mathrm{Sym}(V_1) \otimes \ldots \otimes \mathrm{Sym}(V_n) = \mathrm{Sym}(V_1 + \ldots + V_n)$ is isomorphic to the algebra of polynomials on $(V_1^* + \ldots + V_n^*)$ over F, which has no zero divisors. This implies ϕ is injective because, in this setting, ϕ is equivalent to multiplying certain homogeneous degree $m_1 - 1 + \ldots + m_n - 1$ polynomials by a fixed homogeneous degree n polynomial.

The intertwining property of ϕ is easy to verify using the fact that α generates a trivial representation in $V_1 \otimes \ldots \otimes V_n$. This concludes the proof. $\square$

Corollary 2.2 follows directly from Proposition 2.1.

Corollary 2.2. *Let $V_1, \ldots, V_n$ be finite-dimensional representations of a Lie algebra. If there exists a trivial representation contained in $V_1 \otimes \ldots \otimes V_n$, then there exists an invariant subspace*

$$\mathrm{Sym}^{m_1 - 1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n - 1} V_n \subset \mathrm{Sym}^{m_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{m_n} V_n \text{ for all } m_i \geq 1.$$

For our purposes, we will now focus on $V \otimes V^*$ for a finite-dimensional representation V of a Lie algebra. $V \otimes V^*$ contains the trivial representation. Let V have the basis $\{v_1, v_2, \ldots, v_k\}$, and let V^* be the dual space with corresponding dual basis $\{f_1, f_2, \ldots, f_k\}$. The trivial representation is generated by $\sum_{i=1}^{k} v_i \otimes f_i$.

Using the given bases of V and V^*, we identify the standard basis elements of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$ with pairs of k-tuples such that c_i equals the number of times v_i appears in the basis element and d_j equals the number of times f_j appears in the basis element. The standard basis for $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$ is then given by

$$\{(c_1, \dots, c_k) \times (d_1, \dots, d_l) \mid c_i \in \mathbb{Z}_{\geq 0}, d_j \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k} c_i = n, \sum_{j=1}^{k} d_j = m\}.$$

For $n, m \geq 1$, consider the linear map

$$\rho : \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V^* \to \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$$

with the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \dots \otimes \alpha_{n-1}) \otimes \mathrm{sym}(\beta_1 \otimes \dots \otimes \beta_{m-1}) \\ &\qquad \longmapsto \sum_{i=1}^{k} \mathrm{sym}(\alpha_1 \otimes \dots \otimes \alpha_{n-1} \otimes v_i) \otimes \mathrm{sym}(\beta_1 \otimes \dots \otimes \beta_{m-1} \otimes f_i). \end{aligned}$$

This is the map defined as multiplication by the element $\sum_{i=1}^{k} v_i \otimes f_i$, which generates the trivial representation in $V \otimes V^*$. Proposition 2.1 shows ρ is an injective, intertwining map.

The dual map to ρ (with n and m interchanged) is the linear map

$$\rho^* : \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^* \to \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V^*$$

with the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \dots \otimes \alpha_n) \otimes \mathrm{sym}(\beta_1 \otimes \dots \otimes \beta_m) \\ &\mapsto \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j(\alpha_i) \mathrm{sym}(\alpha_1 \otimes \dots \otimes \hat{\alpha}_i \otimes \dots \otimes \alpha_n) \otimes \mathrm{sym}(\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_m). \end{aligned}$$

ρ^* is a surjective, intertwining map.

For $V \otimes V^*$, Corollary 2.2 can be applied as follows. There exists an invariant subspace

$$\mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{m-1}V^* \subset \mathrm{Sym}^{n}V \otimes \mathrm{Sym}^{m}V^* \text{ for all integers } n, m \geq 1.$$

2.2 The case of $\mathfrak{sp}(4, \mathbb{C})$

We will apply the above result from Corollary 2.2 to representations of the Lie algebra $\mathfrak{sp}(4, \mathbb{C})$, where

$$\mathfrak{sp}(4, \mathbb{C}) = \{A \in \mathfrak{gl}(4, \mathbb{C}) \mid A^t J + JA = 0\} \text{ and } J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Evidently, $\mathfrak{sp}(4, \mathbb{C})$ is 10-dimensional and has the following basis,

$$H_1 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{bmatrix}$$

$$X_{\alpha_1} = \begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&-1\\0&0&0&0 \end{bmatrix} \quad X_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad X_{\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix} \quad X_{\alpha_2} = \begin{bmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

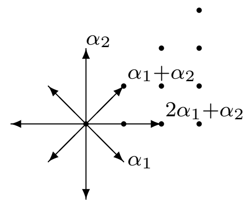
$$Y_{\alpha_1} = \begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&-1&0 \end{bmatrix} \quad Y_{2\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix} \quad Y_{\alpha_1+\alpha_2} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0 \end{bmatrix} \quad Y_{\alpha_2} = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}.$$

In this basis, the simple roots are α_1 and α_2, the Cartan subalgebra is $\mathfrak{h} = \langle H_1, H_2 \rangle$, and for each root α,

$$\mathfrak{s}^\alpha = \mathrm{span}\{X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Any weight (w_1, w_2) can be thought of as the pair of eigenvalues associated to H_1 and H_2, respectively, for the corresponding weight vector. The dominant Weyl chamber is $\{(n, m) \in \mathbb{Z} \times \mathbb{Z} : n \geq m \geq 0\}$. Let $V(n, m)$ be the irreducible representation with highest weight (n, m), $n \geq m$.

The following displays the root system and the first few weights of the dominant Weyl chamber.



The Weyl dimension formula, tailored to our situation, appears in [6], Section 7.6.3. It states that

$$\dim V(n,m) = \frac{1}{6}(n-m+1)(m+1)(n+2)(n+m+3).$$

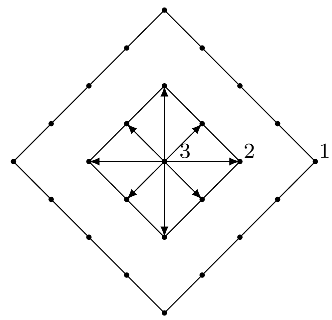
The following table displays $V(1,0)$, the standard representation of $\mathfrak{sp}(4,\mathbb{C})$, for this previously defined basis of $\mathfrak{sp}(4,\mathbb{C})$ and the standard basis of $\mathbb{C}^4$ and its dual representation with corresponding basis $\{f_1, f_2, f_3, f_4\}$. These representations are isomorphic via $f_1 \mapsto -e_4$, $f_2 \mapsto -e_3$, $f_3 \mapsto e_2$, $f_4 \mapsto e_1$, but the different formulas for both of them will be used in subsequent calculations.

	e_1	e_2	e_3	e_4	f_1	f_2	f_3	f_4
H_1	e_1	0	0	$-e_4$	$-f_1$	0	0	f_4
H_2	0	e_2	$-e_3$	0	0	$-f_2$	f_3	0
X_{α_1}	0	e_1	0	$-e_3$	$-f_2$	0	f_4	0
$X_{2\alpha_1+\alpha_2}$	0	0	0	e_1	$-f_4$	0	0	0
$X_{\alpha_1+\alpha_2}$	0	0	e_1	e_2	$-f_3$	$-f_4$	0	0
X_{α_2}	0	0	e_2	0	0	$-f_3$	0	0
Y_{α_1}	e_2	0	$-e_4$	0	0	$-f_1$	0	f_3
$Y_{2\alpha_1+\alpha_2}$	e_4	0	0	0	0	0	0	$-f_1$
$Y_{\alpha_1+\alpha_2}$	e_3	e_4	0	0	0	0	$-f_1$	$-f_2$
Y_{α_2}	0	e_3	0	0	0	0	$-f_2$	0

The weights of $V(1,0)$ are $\{(1,0),(0,1),(0,-1),(-1,0)\}$, and e_1 is a highest weight vector.

It can be easily shown that $V(n,0) = \mathrm{Sym}^n V(1,0)$. First, there is a highest weight vector, $\mathrm{sym}(e_1 \otimes \ldots \otimes e_1)$, in $\mathrm{Sym}^n V(1,0)$ with weight $(n,0)$, and therefore $V(n,0) \subset \mathrm{Sym}^n V(1,0)$. Then using the Weyl dimension formula, $V(n,0)$ has the same dimension as $\mathrm{Sym}^n V(1,0)$ and thus $V(n,0) = \mathrm{Sym}^n V(1,0)$.

The weight diagram for $V(n,0) = \mathrm{Sym}^n V(1,0)$ is a series of nested diamonds with leading weights $(n-2i,0)$ and with multiplicities $i+1$ along the diamonds, $0 \le i \le \lfloor \frac{n}{2} \rfloor$. The following is the weight diagram for $V(4,0)$.



Proposition 2.3. *For $\mathfrak{sp}(4,\mathbb{C})$ and its standard representation $V = V(1,0)$,*

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V = (\mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V) \oplus \bigoplus_{p=0}^{m} V(n+m-p,p)$$

for integers $n \geq m \geq 1$.

Proof. Given $n \geq m$ and using the previously described basis, we define for all integers p such that $0 \leq p \leq m$ the following vector in $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$,

$$\begin{aligned} v_p &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i (n-p+i, p-i, 0, 0) \times (0,0,i,m-i) \\ &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mathrm{sym}(\underbrace{e_1 \otimes \ldots \otimes e_1}_{n-p+i} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{p-i}) \\ &\qquad \otimes \mathrm{sym}(\underbrace{f_3 \otimes \ldots \otimes f_3}_{i} \otimes \underbrace{f_4 \otimes \ldots \otimes f_4}_{m-i}). \end{aligned}$$

This vector is in the kernel of the map ρ^* defined in Section 2.1 because

$$(n-p+i, p-i, 0, 0) \times (0,0,i,m-i) \mapsto 0 + \ldots + 0 = 0.$$

Also, this vector is a highest weight vector with weight

$$(n-p+i)(1,0) + (p-i)(0,1) + i(0,1) + (m-i)(1,0) = (n+m-p,p).$$

To see v_p is a highest weight vector, it is enough to show that it is in the kernel of X_α for any α.

First, the only relevant calculations are $X_\alpha.e_1$, $X_\alpha.e_2$, $X_\alpha.f_3$, and $X_\alpha.f_4$. These will all be equal to zero except for $\alpha = \alpha_1$. Therefore, we only need to show v_p is in the kernel of X_{α_1}. $X_{\alpha_1}.e_1 = X_{\alpha_1}.f_4 = 0$, $X_{\alpha_1}.e_2 = e_1$, and $X_{\alpha_1}.f_3 = f_4$. By definition,

$$\begin{aligned} &X_{\alpha_1}.(n-p+i, p-i, 0, 0) \times (0, 0, i, m-i) \\ &\quad = X_{\alpha_1}.\mathrm{sym}(\underbrace{e_1 \otimes \ldots \otimes e_1}_{n-p+i} \otimes \underbrace{e_2 \otimes \ldots \otimes e_2}_{p-i}) \otimes \mathrm{sym}(\underbrace{f_3 \otimes \ldots \otimes f_3}_{i} \otimes \underbrace{f_4 \otimes \ldots \otimes f_4}_{m-i}). \end{aligned}$$

This becomes

$$\begin{aligned} &(n-p+i)\mathrm{sym}(X_{\alpha_1}.e_1 \otimes e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (p-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes X_{\alpha_1}.e_2 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad\qquad \otimes \mathrm{sym}(X_{\alpha_1}.f_3 \otimes f_3 \otimes \ldots \otimes f_3 \otimes f_4 \otimes \ldots \otimes f_4) \\ &+ (m-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2) \\ &\qquad\qquad \otimes \mathrm{sym}(f_3 \otimes \ldots \otimes f_3 \otimes X_{\alpha_1}.f_4 \otimes f_4 \otimes \ldots \otimes f_4). \end{aligned}$$

This is equal to $(n-p+i)(0) + (p-i)(n-p+i+1, p-i-1, 0, 0) \times (0, 0, i, m-i) + (i)(n-p+i, p-i, 0, 0) \times (0, 0, i-1, m-i+1) + (m-i)(0)$ (with the understanding that when $i = p$ there is no second term and when $i = 0$ there is no third term here). From here $X_{\alpha_1}.v_p = 0$ is a straightforward calculation.

For each of the highest weight vectors, v_p, with weight $(n+m-p,p)$ and in the kernel of ρ^*, there is an irreducible representation $V(n+m-p,p)$ contained in the kernel. Since all of the weights $\{(n+m-p,p) : 0 \leq p \leq m\}$, are distinct,

$$\bigoplus_{p=0}^{m} V(n+m-p,p) \subset \ker(\rho^*).$$

It follows from semisimplicity and the surjectivity of ρ^* that

$$\begin{aligned}(\mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{m-1}V^*) \oplus \bigoplus_{p=0}^{m} V(n+m-p,p) \\ \subset (\mathrm{Sym}^{n-1}V \otimes \mathrm{Sym}^{m-1}V^*) \oplus \ker(\rho^*) \\ = \mathrm{Sym}^{n}V \otimes \mathrm{Sym}^{m}V^*\end{aligned}$$

for $n \geq m \geq 1$. The Weyl dimension formula shows that this inclusion is actually an equality. Note that V^* can be replaced by V since this representation is self-dual. □

Note that all of the highest weight vectors in $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V$, $V = V(1,0)$, can be determined using the proof of Proposition 2.3, the map ρ from Section 2.1, and the isomorphism between the standard representation and its dual.

In [10], Littelmann provides a generalization of the Littlewood-Richardson rule in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. This generalization provides an algorithm for decomposing tensor products of irreducible representations using Young tableaux and can be utilized to produce the result of Proposition 2.3.

Corollary 2.4. *For integers $n \geq m = 1$,*

$$V(n,0) \otimes V(1,0) = V(n+1,0) \oplus V(n,1) \oplus V(n-1,0)$$

For $n \geq m \geq 2$,

$$\begin{aligned}(V(n,0) &\otimes V(m,0)) \oplus (V(n,0) \otimes V(m-2,0)) \\ &= (V(n+1,0) \otimes V(m-1,0)) \oplus V(n,m) \oplus (V(n-1,0) \otimes V(m-1,0))\end{aligned}$$

Proof. Recall $\mathrm{Sym}^n V(1,0) = V(n,0)$. The first assertion is the special case of Proposition 2.3 where $m = 1$. Using Proposition 2.3, when $n \geq m \geq 2$,

$$V(n,0) \otimes V(m,0) = (V(n-1,0) \otimes V(m-1,0)) \oplus \bigoplus_{p=0}^{m} V(n+m-p,p)$$

and

$$V(n+1,0) \otimes V(m-1,0) = (V(n,0) \otimes V(m-2,0)) \oplus \bigoplus_{p=0}^{m-1} V(n+m-p,p).$$

Combining these equations yields the assertion. □

In the Grothendieck group of all representations of $\mathfrak{sp}(4,\mathbb{C})$, for $V = V(1,0)$, we get

$$
\begin{aligned}
V(n,0) &= \mathrm{Sym}^n V && n \geq 0 \\
V(n,1) &= \mathrm{Sym}^n V \otimes V - \mathrm{Sym}^{n+1} V - \mathrm{Sym}^{n-1} V && n \geq 1 \\
V(n,m) &= \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V + \mathrm{Sym}^n V \otimes \mathrm{Sym}^{m-2} V && n \geq m \geq 2 \\
&\quad - \mathrm{Sym}^{n-1} V \otimes \mathrm{Sym}^{m-1} V - \mathrm{Sym}^{n+1} V \otimes \mathrm{Sym}^{m-1} V.
\end{aligned}
$$

This result can also be derived in a less elementary way from a proposition in Section 24.2 in [5], which gives a formula for the character of an irreducible representation of a simplectic Lie algebra in terms of the characters of symmetric powers of the standard representation.

2.3 Weight multiplicities in $V(n,0) \otimes V(m,0)$

Since any irreducible representation of $\mathfrak{sp}(4,\mathbb{C})$ can be written as a formal combination of tensor products of symmetric powers of the standard representation, the problem of determining weight multiplicities in an irreducible representation is reduced to the problem of determining weight multiplicities in $V(n,0)\otimes V(m,0)$. We will now begin a combinatorics argument, which will produce an explicit formula for the weight multiplicities of $V(n,0) \otimes V(m,0)$.

Using previous notation, the set

$$
\{(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4) | c_i, d_j \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{4} c_i = n, \sum_{j=1}^{4} d_j = m\}
$$

is a basis of weight vectors for $V(n,0) \otimes V(m,0)$. The weight of $(c_1,c_2,c_3,c_4) \times (d_1,d_2,d_3,d_4)$ is $((c_1+d_1)-(c_4+d_4),(c_2+d_2)-(c_3+d_3))$. The only dominant weights of $V(n,0) \otimes V(m,0)$ with a nonzero multiplicity are of the form $(n+m-2i-j, j)$ for $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$. To determine the multiplicity of a dominant weight, we need only count the number of distinct vectors of the form $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$ with that weight.

In other words, the multiplicity of the dominant weight $(n+m-2i-j, j)$ is equal to the number of distinct vectors $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$ such that $(c_1+d_1)-(c_4+d_4) = n+m-2i-j$ and $(c_2+d_2)-(c_3+d_3) = j$. Let $x_r = c_r + d_r$. Solving the system

$$x_1 - x_4 = n + m - 2i - j$$
$$x_2 - x_3 = j$$

yields the solution set satisfying

$$x_1 = n + m - 2i - j + x_4$$
$$x_2 = i + j - x_4$$
$$x_3 = i - x_4$$

for $x_4 \in \mathbb{Z}$ and $0 \leq x_4 \leq i$.

For a fixed $x = x_4$, the number of distinct vectors $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$ such that $(c_1+d_1)-(c_4+d_4) = n+m-2i-j$ and $(c_2+d_2)-(c_3+d_3) = j$ is equivalent to the number of distinct ways to find (d_1, d_2, d_3, d_4) such that $\sum_{r=1}^{4} d_r = m$ and $0 \leq d_r \leq x_r$ for any r. Since $x_3 + x_4 = i$ and $x_1 + x_2 = n + m - i$, we can fix an integer k such that $0 \leq k \leq \min(m, i)$, and the number of distinct ways to find (d_1, d_2, d_3, d_4) with the desired conditions is equivalent to $\sum_{k=0}^{\min(m,i)} f(x,k) * g(x,k)$, where $f(x,k)$ is the number of distinct ways to find (d_1, d_2) such that $d_1 + d_2 =$

$m-k$ and $0 \le d_r \le x_r$ for $r = 1, 2$ and where $g(x, k)$ is the number of distinct ways to find (d_3, d_4) such that $d_3 + d_4 = k$ and $0 \le d_r \le x_r$ for $r = 3, 4$. With these definitions,

$$f(x, k) = \min(n + m - 2i - j + x + 1, i + j - x + 1, n + k - i, m - k + 1)$$
$$g(x, k) = \min(x + 1, i - x + 1, k + 1, i - k + 1).$$

The multiplicity of $(n+m-2i-j, j)$ in $V(n, 0) \otimes V(m, 0)$, for $0 \le i \le \lfloor \frac{n+m}{2} \rfloor$ and $0 \le j \le \lfloor \frac{n+m}{2} \rfloor - i$, is

$$M(n + m - 2i - j, j) = \sum_{x=0}^{i} \sum_{k=0}^{\min(m,i)} f(x, k) * g(x, k).$$

Theorem 2.5. *The multiplicity of the dominant weight* $(n + m - 2i - j, j)$*,* $0 \le i \le \lfloor \frac{n+m}{2} \rfloor$ *and* $0 \le j \le \lfloor \frac{n+m}{2} \rfloor - i$*, in the representation* $V(n, 0) \otimes V(m, 0)$ *of* $\mathfrak{sp}(4, \mathbb{C})$ *is given in the following table. The conditions on* $n, m, i,$ *and* j *are in the first two columns, and the third column is the corresponding multiplicity.*

$n \geq 2i+j,$	$m \geq 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2)$
	$i+j \leq m \leq 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2)$ $-R(\beta)$
	$j \leq m \leq i+j, m \geq i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2)$ $-R(\gamma)$
	$m \leq j, m \geq i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2)$
	$j \leq m \leq i+j, m \leq i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2)$ $-R(\gamma)$
	$m \leq j, m \leq i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2)$
$n \leq 2i+j,$	$i+j \leq m \leq 2i+j$	$\frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2)$ $-R(\alpha)-R(\beta)$
	$j \leq m \leq i+j, m \geq i$	$\frac{1}{12}(i+1)(i+2)(i+3)(2m-i+2)$ $-R(\alpha)-R(\gamma)$
	$j \leq m \leq i+j, m \leq i$	$\frac{1}{12}(m+1)(m+2)(m+3)(2i-m+2)$ $-R(\alpha)-R(\gamma)$

In the table, $\alpha = 2i+j-n, \beta = 2i+j-m, \gamma = m-j,$ *and* $R(z)$ *is defined as*

$$R(z) = \begin{cases} \frac{1}{48} z(z+2)^2(z+4) & z \text{ even} \\ \frac{1}{48}(z+1)(z+3)(z^2+4z+1) & z \text{ odd.} \end{cases}$$

Proof. This is a direct result of

$$M(n+m-2i-j,j) = \sum_{x=0}^{i} \sum_{k=0}^{\min(m,i)} f(x,k) * g(x,k).$$

To use this definition to compute the multiplicity, consider

$$M(n+m-2i-j,j) = \sum_{k=0}^{\min(m,i)} S(k)$$

for $S(k) = \sum_{x=0}^{i} f(x,k) * g(x,k)$. The definitions of f and g then produce different cases. We will show one case as an example.

For $n \geq 2i+j$ and $m \geq 2i+j$,

$$\min(n+m-2i-j+x+1, i+j-x+1, n+k-i, m-k+1) = i+j-x+1,$$

and $S(k) = \sum_{x=0}^{i}(i+j-x+1) * g(x,k)$. To determine $S(k)$, we wish to sum over $x = 0, \ldots, i$ given a particular k. We will separately consider the cases when $k < \frac{i}{2}$, $k = \frac{i}{2}$, and $k > \frac{i}{2}$.

Assume $k < \frac{i}{2}$. When $x \leq k$, $g(k,x) = x+1$. When $k \leq x \leq i-k$, $g(k,x) = k+1$. When $x \geq i-k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) = &\sum_{x=0}^{k}(i+j-x+1)(x+1) + \sum_{x=k}^{i-k}(i+j-x+1)(k+1) \\ &+ \sum_{x=i-k}^{i} (i+j-x+1)(i-x+1) \\ &- (i+j-k+1)(k+1) - (i+j-(i-k)+1)(k+1) \\ = &\frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Assume $k = \frac{i}{2}$. When $x \le k$, $g(k,x) = x+1$. When $x \ge k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) &= \sum_{x=0}^{k}(i+j-x+1)(x+1) + \sum_{x=k}^{i}(i+j-x+1)(i-x+1) \\ &\quad - (i+j-k+1)(k+1) \\ &= \frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Finally, assume $k > \frac{i}{2}$. When $x \le i-k$, $g(k,x) = x+1$. When $i-k \le x \le k$, $g(k,x) = i-k+1$. When $x \ge k$, $g(k,x) = i-x+1$.

$$\begin{aligned} S(k) &= \sum_{x=0}^{i-k}(i+j-x+1)(x+1) + \sum_{x=i-k}^{k}(i+j-x+1)(i-k+1) \\ &\quad + \sum_{x=k}^{i}(i+j-x+1)(i-x+1) \\ &\quad - (i+j-(i-k)+1)(i-k+1) - (i+j-k+1)(i-k+1) \\ &= \frac{1}{2}(k+1)(i-k+1)(i+2j+2). \end{aligned}$$

Since $S(k) = \frac{1}{2}(k+1)(i-k+1)(i+2j+2)$ for any $k = 0, \ldots, i$,

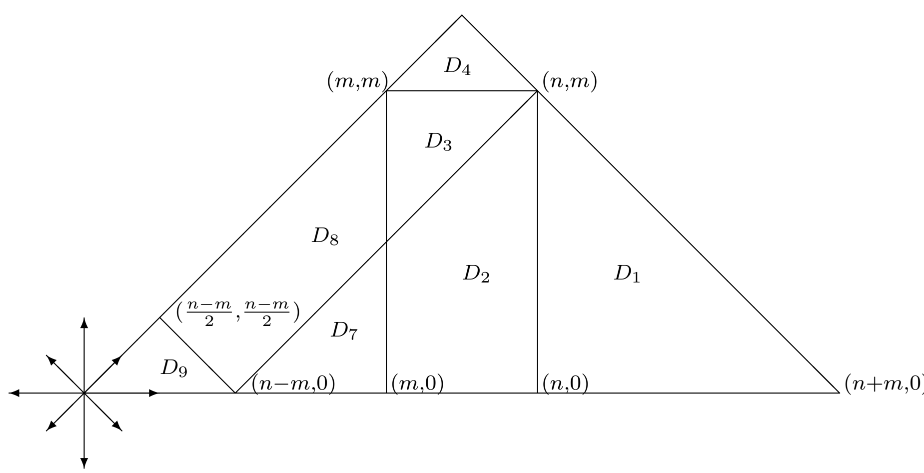
$$\begin{aligned} M(n+m-2i-j, j) &= \sum_{k=0}^{i} \frac{1}{2}(k+1)(i-k+1)(i+2j+2) \\ &= \frac{1}{12}(i+1)(i+2)(i+3)(i+2j+2). \end{aligned}$$

$\square$

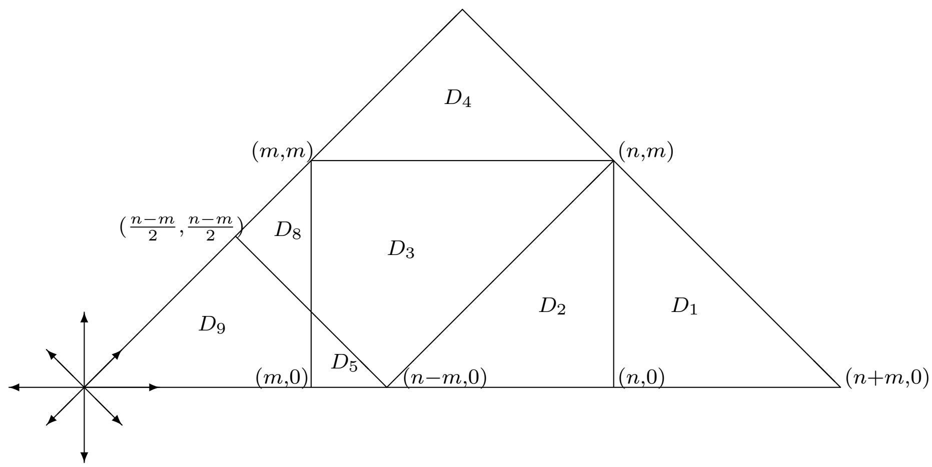
Index the conditions on n, m, i, and j as follows.

$$\begin{array}{llr}
n \geq 2i+j & m \geq 2i+j & D_1 \\
 & i+j \leq m \leq 2i+j & D_2 \\
 & j \leq m \leq i+j, m \geq i & D_3 \\
 & m \leq j, m \geq i & D_4 \\
 & j \leq m \leq i+j, m \leq i & D_5 \\
 & m \leq j, m \leq i & D_6 \\
n \leq 2i+j & i+j \leq m \leq 2i+j & D_7 \\
 & j \leq m \leq i+j, m \geq i & D_8 \\
 & j \leq m \leq i+j, m \leq i & D_9
\end{array}$$

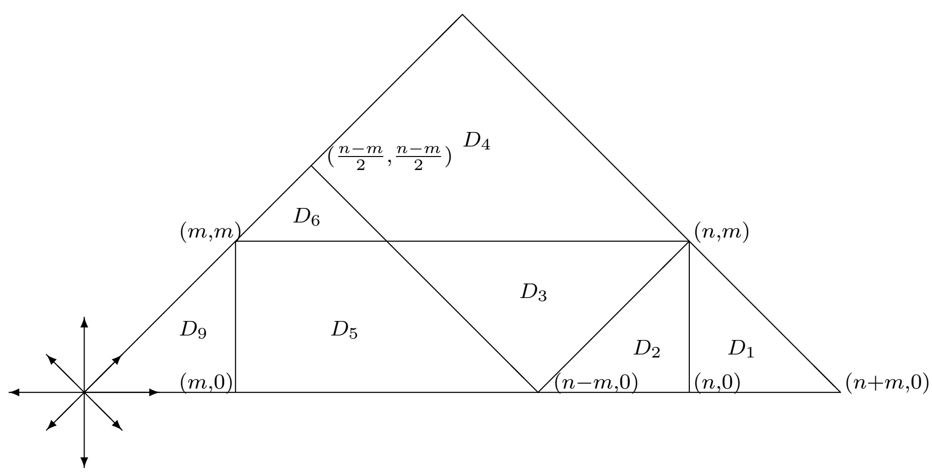
These conditions on the dominant weights of $V(n,0) \otimes V(m,0)$ create separate sections D_i. There are three main cases of this. The following diagrams display these cases. In the first case, $n \leq 2m$.



In the second case, $2m \leq n \leq 3m$.



In the third case, $n \geq 3m$.



The multiplicity of a weight lying on a line can be calculated using the formula for any section sharing that line as an edge.

In the boundary cases, such as when $m = 0$, $n = m$, $n = 2m$, or $n = 3m$, there will be fewer sections, but the sectioning of the triangle of dominant weights can still be derived from the main cases. For example, when $m = 0$ the triangle of dominant weights will only contain D_6, and when $n = m$ the triangle of dominant weights will be split down the middle into the sections D_7 and D_1.

2.4 Weight multiplicities in $V(n,m)$

Using the results of Section 3 and Section 4, the multiplicities of the weights in the representation $V(n,m)$ can be determined. Let $M(n+m-2i-j,j)(V)$ be the multiplicity of the dominant weight $(n+m-2i-j,j)$, $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$, in the representation V. The results of Corollary 2.4 can be applied to weight multiplicities. The multiplicity of the weight $(n+m-2i-j,j)$ for $m=0$, $m=1$, and $m \geq 2$ can be found from the following identities, keeping in mind that $(n+m-2i-j,j) = ((n+1)+(m-1)-2i-j,j) = (n+(m-2)-2(i-1)-j,j) = ((n-1)+(m-1)-2(i-1)-j,j)$ and when $i=0$, any $M(n'+m'-2(i-1)-j,j)(V(n',0) \otimes V(m',0)) = 0$.

$$
\begin{aligned}
&M(n-2i-j,j)(V(n,0)) \\
&\qquad = M(n-2i-j,j)(V(n,0)\otimes V(0,0)) \\
&M(n+1-2i-j,j)(V(n,1)) \\
&\qquad = M(n+1-2i-j,j)(V(n,0)\otimes V(1,0)) \\
&\qquad\quad - M(n+1-2i-j,j)(V(n+1,0)\otimes V(0,0)) \\
&\qquad\quad - M(n-1-2(i-1)+j,j)(V(n-1,0)\otimes V(0,0)) \\
&M(n+m-2i-j,j)(V(n,m)) \\
&\qquad = M(n+m-2i-j,j)(V(n,0)\otimes V(m,0)) \\
&\qquad\quad + M(n+m-2-2(i-1)-j,j)(V(n,0)\otimes V(m-2,0)) \\
&\qquad\quad - M(n+m-2-2(i-1)-j,j)(V(n-1,0)\otimes V(m-1,0)) \\
&\qquad\quad - M(n+m-2i-j,j)(V(n+1,0)\otimes V(m-1,0)).
\end{aligned}
$$

Combining these results with Theorem 2.5 gives a closed formula for the weight multiplicities of the dominant weights of $V(n,m)$.

Theorem 2.6. *The multiplicity of the dominant weight* $(n+m-2i-j,j)$, $0 \le i \le \lfloor \frac{n+m}{2} \rfloor$ *and* $0 \le j \le \lfloor \frac{n+m}{2} \rfloor - i$, *in the representation* $V(n,m)$ *of* $\mathfrak{sp}(4,\mathbb{C})$ *is given in the following table. The conditions on* n, m, i, *and* j *are in the first two columns, and the third column is the corresponding multiplicity.*

$n > 2i+j$	$m > 2i+j$	0
$n \geq 2i+j$	$m = 2i+j$	1
	$i+j \leq m \leq 2i+j$	$P(\beta)$
	$j \leq m \leq i+j, m \geq i$	$\frac{1}{2}(i+1)(i+2) - Q(\gamma)$
	$m \leq j, m \geq i$	$\frac{1}{2}(i+1)(i+2)$
	$j \leq m \leq i+j, m \leq i$	$\frac{1}{2}(2i-m+2)(m+1) - Q(\gamma)$
	$m \leq j, m \leq i$	$\frac{1}{2}(2i-m+2)(m+1)$
$n \leq 2i+j$	$i+j \leq m \leq 2i+j$	$P(\beta) - Q(\alpha)$
	$j \leq m \leq i+j, m \geq i$	$\frac{1}{2}(i+1)(i+2) - Q(\alpha) - Q(\gamma)$
	$j \leq m \leq i+j, m \leq i$	$\frac{1}{2}(2i-m+2)(m+1) - Q(\alpha) - Q(\gamma)$

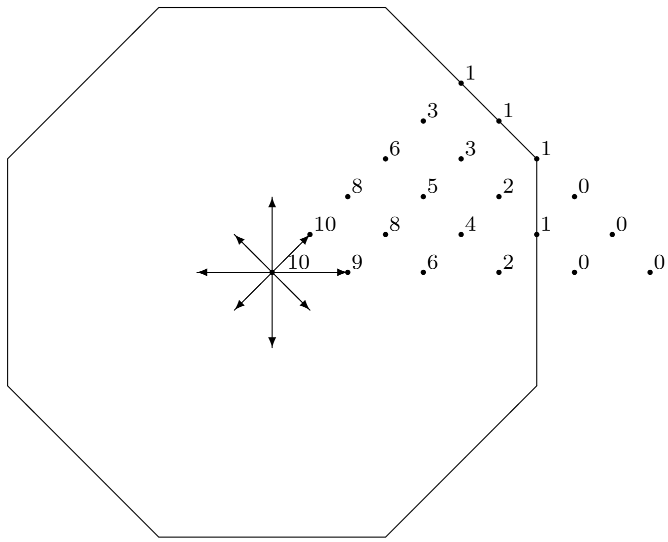
In the table, $\alpha = 2i+j-n, \beta = 2i+j-m, \gamma = m-j,$ $P(z)$ *and* $Q(z)$ *are defined as*

$$P(z) = \begin{cases} \frac{1}{4}(z+2)^2 & z \text{ even} \\ \frac{1}{4}(z+1)(z+3) & z \text{ odd} \end{cases}$$

$$Q(z) = \begin{cases} \frac{1}{4}z(z+2) & z \text{ even} \\ \frac{1}{4}(z+1)^2 & z \text{ odd.} \end{cases}$$

The multiplicities of all other weights can be determined through reflections. It is also easy enough to check that these multiplicities coincide with the multiplicity formula found at the end of [3].

This picture is of the multiplicities of the dominant weights $(n+m-2i-j, j)$ in $V(7,3)$.



The following are a few examples of the calculations required to determine the multiplicities of these weights using Theorem 2.6.

The weight $(8,2) = (10-2(0)-2, 2)$, where $i = 0$ and $j = 2$. Then $n = 7 > 2 = 2i + j$ and $m = 3 > 2 = 2i + j$. Therefore $M(8,2) = 0$.

The weight $(6,0) = (10-2(2)-0, 0)$, where $i = 2$ and $j = 0$. Then $n = 7 > 4 = 2i + j$ and $2 = i + j < m = 3 < 2i + j = 4$. Here, $\beta = 2i + j - m = 4 - 3 = 1$. Therefore $M(6,0) = \frac{1}{4}(1+1)(3+1) = 2$.

The weight $(3,1) = (10-2(3)-1, 1)$, where $i = 3$ and $j = 1$. Then $n = 7 = 2i+j$, $1 = j < m = 3 < i + j = 4$, and $m = i = 3$. Here, $\gamma = m - j = 3 - 1 = 2$. Therefore $M(3,1) = \frac{1}{2}(3+1)(3+2) - \frac{1}{4}(2)(2+2) = 8$.

The weight $(0,0) = (10-2(5)-0, 0)$, where $i = 5$ and $j = 0$. Then $n =$

$7 < 10 = 2i + j$, $0 = j < m = 3 < i + j = 5$, and $m < i = 5$. Here, $\alpha = 2i + j - n = 3$ and $\gamma = m - j = 3 - 0 = 3$. Therefore $M(0,0) = \frac{1}{2}(2 * 5 - 3 + 2)(3 + 1) - \frac{1}{4}(3 + 1)^2 - \frac{1}{4}(3 + 1)^2 = 10$.

Chapter 3

L- and ε-factors for $\mathrm{Sp}(4)$

3.1 The real Weil group

As in [15], the real Weil group is $W_{\mathbb{R}} = \mathbb{C}^{\times} \sqcup j\mathbb{C}^{\times}$ such that $j^2 = -1$ and $jzj^{-1} = \bar{z}$. The L- and ε- factors for Sp(4) are calculated for a particular representation of the real Weil group $W_{\mathbb{R}}$, $\hat{\mu} \circ \zeta : W_{\mathbb{R}} \to \mathrm{GL}(V)$, where $\zeta : W_{\mathbb{R}} \to \mathrm{Sp}(4)$ is a fixed representation of $W_{\mathbb{R}}$ and $\hat{\mu}$ is a representation of Sp(4), $\hat{\mu} : \mathrm{Sp}(4) \to \mathrm{GL}(V)$. The fixed representation of the real Weil group, $\zeta : W_{\mathbb{R}} \to \mathrm{Sp}(4)$, is defined such that

$$re^{i\theta} \mapsto \begin{bmatrix} e^{ik\theta} & & & \\ & e^{il\theta} & & \\ & & e^{-il\theta} & \\ & & & e^{-ik\theta} \end{bmatrix}$$

and

$$j \mapsto \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} = J,$$

where l and k are both odd integers, so that $j^2 = -1 \mapsto -1 = J^2$.

In [8], the representation theory of $W_{\mathbb{R}}$ includes the following facts. Every finite-dimensional representation of $W_{\mathbb{R}}$ is completely reducible, and every irreducible representation is either one- or two-dimensional. Furthermore, every one-dimensional representation is of the form $\phi_{+,t}$ or $\phi_{-,t}$,

$$\phi_{+,t} : z \mapsto |z|^{2t}, j \mapsto 1$$

or

$$\phi_{+,t} : z \mapsto |z|^{2t}, j \mapsto -1,$$

and every two-dimensional representation is of the form $\phi_{p,t}$ for some integer p,

$$\phi_{p,t} : re^{i\theta} \mapsto \begin{bmatrix} r^{2t}e^{ip\theta} & \\ & r^{2t}e^{-ip\theta} \end{bmatrix}, j \mapsto \begin{bmatrix} & (-1)^p \\ 1 & \end{bmatrix}.$$

Note that $\phi_{p,t} \cong \phi_{-p,t}$ and when $p = 0$, $\phi_{p,t}$ decomposes into $\phi_{+,t} \oplus \phi_{-,t}$. Since $W_{\mathbb{R}}$ has the complete reducibility property, any representation of $W_{\mathbb{R}}$ can be written as $\phi = \oplus\phi_i$ for some irreducible representations ϕ_i, and $L(s,\phi) = \Pi_i L(s,\phi_i)$ and $\varepsilon(s,\phi) = \Pi_i \varepsilon(s,\phi_i)$. Therefore we only need to know $L(s,\phi)$ and $\varepsilon(s,\phi)$ for irreducible representation ϕ, and then we can calculate the L- and ε-factors for any representation. The following table displays these factors for the irreducible representations of $W_{\mathbb{R}}$.

ϕ	$L(s,\phi)$	$\varepsilon(s,\phi)$
$\phi_{+,t}$	$\Gamma_{\mathbb{R}}(s+t)$	1
$\phi_{-,t}$	$\Gamma_{\mathbb{R}}(s+t+1)$	i
$\phi_{p,t}$	$\Gamma_{\mathbb{C}}(s+t+p/2)$	i^{l+1}

In this table, $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$. Also, Legendre's formula

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\pi^{1/2}\Gamma(s)$$

produces the following equality,

$$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s).$$

This means that the L- and ε-factors of $\phi_{0,t} = \phi_{+,t} \oplus \phi_{-,t}$ can be calculated using the definitions for $\phi_{p,t}$ and setting $p = 0$ because these definitions are equivalent to $L(s,\phi_{+,t})L(s,\phi_{-,t})$ and $\varepsilon(s,\phi_{+,t})\varepsilon(s,\phi_{-,t})$, respectively, when $p = 0$. Also, for our purposes given our definiton of ζ, t will always be equal to zero. Therefore, we will omit this parameter from now on.

The question of calculating L- and ε-factors for a representation becomes a question of what is the decomposition of that representation into one- and two-dimensional representations of $W_{\mathbb{R}}$.

3.2 Archimedean factors of $\mathrm{Sp}(4)$

Since $\mathrm{Sp}(4)$ has the complete reducibility property, any representation of $\mathrm{Sp}(4)$, $\hat{\mu} = \oplus\hat{\mu}_i$ for irreducible representations $\hat{\mu}_i$. Then for the representation of $W_{\mathbb{R}}$, $\hat{\mu}\circ\zeta : W_{\mathbb{R}} \to \mathrm{GL}(V)$, where $\zeta : W_{\mathbb{R}} \to \mathrm{Sp}(4)$ is a fixed representation of $W_{\mathbb{R}}$, $\hat{\mu}\circ\zeta = \oplus(\hat{\mu}_i\circ\zeta)$. The L- and ε-factors corresponding to this representation of $\mathrm{Sp}(4)$ and the fixed representation of $W_{\mathbb{R}}$ will be the product of the L- and ε-factors corresponding to the $\hat{\mu}_i\circ\zeta$. Therefore, we only need to determine the L- and ε-factors of the $\hat{\mu}\circ\zeta$ for the irreducible representations $\hat{\mu}$ of $\mathrm{Sp}(4)$ to be able to calculate the factors for any representation of $\mathrm{Sp}(4)$.

Any representation $\hat{\mu} : \mathrm{Sp}(4) \to \mathrm{GL}(V)$ is in one-to-one correspondence with $\mu : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{gl}(V)$ via the exponential map, $\mathfrak{gl}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$. We can apply the results of Section 2.2 to $\mathrm{Sp}(4)$ and interpret them in terms of L- and ε- factors. From Corollary 2.4,

$$\begin{aligned}
V(n,0) &= \mathrm{Sym}^n V && n \geq 0\\
V(n,1) &= \mathrm{Sym}^n V\otimes V - \mathrm{Sym}^{n+1}V - \mathrm{Sym}^{n-1}V && n \geq 1\\
V(n,m) &= \mathrm{Sym}^n V\otimes \mathrm{Sym}^m V + \mathrm{Sym}^n V\otimes \mathrm{Sym}^{m-2}V && n\geq m\geq 2\\
&\quad - \mathrm{Sym}^{n-1}V\otimes\mathrm{Sym}^{m-1}V - \mathrm{Sym}^{n+1}V\otimes \mathrm{Sym}^{m-1}V.
\end{aligned}$$

Now, let $L(s,V)$ be L-factor of the representation $\mu\circ\zeta : W_{\mathbb{R}} \to \mathrm{GL}(V)$ for the representation V of Sp(4),

$$L(s,V(n,1)) = \frac{L(s,V(n,0)\otimes V(1,0))}{L(s,V(n+1,0))L(s,V(n-1,0))}$$

for $n \geq 1$,

$$L(s,V(n,m)) = \frac{L(s,V(n,0)\otimes V(m,0))L(s,V(n,0)\otimes V(m-2,0))}{L(s,V(n-1,0)\otimes V(m-1,0))L(s,V(n+1,0)\otimes V(m-1,0))}$$

for $n \geq m \geq 2$.

Let $\varepsilon(s,V)$ be ε-factor of the representation $\mu\circ\zeta : W_{\mathbb{R}} \to \mathrm{GL}(V)$ for the representation V of Sp(4),

$$\varepsilon(s,V(n,1)) = \frac{\varepsilon(s,V(n,0)\otimes V(1,0))}{\varepsilon(s,V(n+1,0))\varepsilon(s,V(n-1,0))}$$

for $n \geq 1$,

$$\varepsilon(s,V(n,m)) = \frac{\varepsilon(s,V(n,0)\otimes V(m,0))\varepsilon(s,V(n,0)\otimes V(m-2,0))}{\varepsilon(s,V(n-1,0)\otimes V(m-1,0))\varepsilon(s,V(n+1,0)\otimes V(m-1,0))}$$

for $n \geq m \geq 2$.

To determine the L- and ε-factors corresponding to Sp(4), it is enough to determine the archimedean factors of the representations $V(n,0)\otimes V(m,0)$.

For $\mu : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{gl}(V)$, let $v \in W_{(a,b)}$ be the weight space with weight (a,b). Then

$$\hat{\mu}(\begin{bmatrix} x & & & \\ & y & & \\ & & y^{-1} & \\ & & & x^{-1} \end{bmatrix})v = x^a y^b v,$$

and the interesting question is what does $\hat{\mu}(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix})v$ equal? Consider the following calculation.

$$\hat{\mu}(\begin{bmatrix} x & & & \\ & y & & \\ & & y^{-1} & \\ & & & x^{-1} \end{bmatrix})\hat{\mu}(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix})v$$

$$= \hat{\mu}(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix})\hat{\mu}(\begin{bmatrix} x^{-1} & & & \\ & y^{-1} & & \\ & & y & \\ & & & x \end{bmatrix})v$$

$$= x^{-a}y^{-b}\hat{\mu}(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix})v$$

This shows that $\hat{\mu}(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix})v \in W_{(-a,-b)}$. If $(a,b) \neq (0,0)$, each $v \in W_{(a,b)}$ pairs with $Jv \in W_{(-a,-b)}$ to generate a two-dimensional representation, ϕ_{ak+bl}, contained in the representation of $W_{\mathbb{R}}$, $\mu \circ \zeta$. If $(a,b) = (0,0)$, each $v \in W_{(0,0)}$ generates a one-dimensional representation contained in the representation $\mu \circ \zeta$ because the only irreducible representations of the real Weil group where $re^{i\theta}$ acts trivially on v are the one-dimensional representations. These calculations are true for weight spaces in any representation.

Now consider $V(n,0) \otimes V(m,0) = \mathrm{Sym}^n V \otimes \mathrm{Sym}^m V$ for V the standard representation. Let π be the standard representation for $\mathfrak{sp}(4,\mathbb{C})$ and let Π be the representation guaranteed for Sp(4) such that the following diagram is commutative, where exp is the normal exponential mapping of matrices.

$$\begin{array}{ccc} \mathrm{Sp}(4) & \xrightarrow{\Pi} & \mathrm{GL}(4,\mathbb{C}) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{sp}(4,\mathbb{C}) & \xrightarrow{\pi} & \mathfrak{gl}(4,\mathbb{C}) \end{array}$$

Here, $\Pi(\exp(X)) = \exp(\pi(X))$ for any $X \in \mathfrak{sp}(4,\mathbb{C})$. Since $\pi(X) = X$ for the standard representation, $\Pi(\exp(X)) = \exp(X)$. This means Π will also have the standard representation. Consequently, the representation

$$\mu : \mathfrak{sp}(4,\mathbb{C}) \to \mathfrak{gl}(\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V)$$

will correspond to the representation of Sp(4),

$$\hat{\mu} : \mathrm{Sp}(4) \to \mathrm{GL}(\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V)$$

such that

$$\hat{\mu}(A)(\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_n) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_m))$$
$$= \mathrm{sym}(A\alpha_1 \otimes \ldots \otimes A\alpha_n) \otimes \mathrm{sym}(A\beta_1 \otimes \ldots \otimes A\beta_m)$$

where Av is just matrix multiplication.

Using previous notation, any standard basis element of $V(n,0) \otimes V(m,0)$ can be written as some pure tensor of standard basis elements of $\mathbb{C}^4$ in the form of $\mathrm{sym}(\alpha) \otimes \mathrm{sym}(\beta) = (c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$, where c_i equals the number of times e_i appears in α and d_j equals the number of times e_j appears in β.

$$Je_1 = -e_4$$
$$Je_2 = -e_3$$
$$Je_3 = e_2$$
$$Je_4 = e_1$$

Therefore, J applied to $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$ equals

$$(-1)^{c_1+c_2+d_1+d_2}(c_4, c_3, c_2, c_1) \times (d_4, d_3, d_2, d_1).$$

Assume $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) \in W_{(0,0)}$. Equating the weight of the vector $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4)$ with $(0,0)$ yields the equations $c_1 + d_1 - (c_4 + d_4) =$

$c_2 + d_2 - (c_3 + d_3) = 0$, so

$$(-1)^{c_1+c_2+d_1+d_2}(c_4, c_3, c_2, c_1) \times (d_4, d_3, d_2, d_1)$$

equals

$$(-1)^{c_3+d_3+c_4+d_4}(c_4, c_3, c_2, c_1) \times (d_4, d_3, d_2, d_1).$$

The weight $(0, 0) = (n + m - 2i - j, j)$ for $i = \frac{n+m}{2}$ and $j = 0$. Since $\frac{n+m}{2}$ is an integer, n and m must have the same parity. For any $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) \in W_{(n+m-2i-j,j)}$, $c_3 + d_3 + c_4 + d_4 = i$ as noted in Section 2.3. Therefore, if $(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) \in W_{(0,0)}$, $c_3 + d_3 + c_4 + d_4 = i = \frac{n+m}{2}$, and $\frac{n+m}{2}$ is even if n and m are both even, and $\frac{n+m}{2}$ is odd if n and m are both odd.

If

$$(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) = (c_1, c_2, c_2, c_1) \times (d_1, d_2, d_2, d_1),$$

$(c_1, c_2, c_2, c_1) \times (d_1, d_2, d_2, d_1)$ generates the representation ϕ_+ when $\frac{n+m}{2}$ is even, and it generates the representation ϕ_- when $\frac{n+m}{2}$ is odd. But for this vector, $2c_1 + 2c_2 = n$ and $2d_1 + 2d_2 = m$, which only happens when n and m are both even. The number of possible vectors, $(c_1, c_2, c_2, c_1) \times (d_1, d_2, d_2, d_1)$, is equal to the number of ways to write $c_1 + c_2 = \frac{n}{2}$ such that $0 \leq c_r \leq \frac{n}{2}$ multiplied by the number of ways to write $d_1 + d_2 = \frac{m}{2}$ such that $0 \leq d_r \leq \frac{m}{2}$. This number is $(\frac{n}{2} + 1)(\frac{m}{2} + 1)$. Therefore when n and m are both even, the vectors $(c_1, c_2, c_2, c_1) \times (d_1, d_2, d_2, d_1)$ of $W_{(0,0)}$ generate $(\frac{n}{2} + 1)(\frac{m}{2} + 1)$ copies of ϕ_+ when $\frac{n+m}{2}$ is even and $(\frac{n}{2} + 1)(\frac{m}{2} + 1)$ copies of ϕ_- when $\frac{n+m}{2}$ is odd.

If

$$(c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) \neq (c_1, c_2, c_2, c_1) \times (d_1, d_2, d_2, d_1),$$

the representation

$$< (c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) + (c_4, c_3, c_2, c_1) \times (d_4, d_3, d_2, d_1) >$$
$$+ < (c_1, c_2, c_3, c_4) \times (d_1, d_2, d_3, d_4) - (c_4, c_3, c_2, c_1) \times (d_4, d_3, d_2, d_1) >$$

equals the representation $\phi_+ \oplus \phi_-$.

For the weight $(0,0)$ with $i = \frac{n+m}{2}$ and $j = 0$, $n \leq 2i + j = n + m$, $0 = j \leq m \leq i + j = \frac{n+m}{2}$, and $m \leq i = \frac{n+m}{2}$. Using Theorem 2.5, the multiplicity of $(0,0)$ is $\frac{1}{12}(m+1)(m+2)(m+3)(n+2) - \frac{1}{24}m(m+2)^2(m+4)$ if m is even and $\frac{1}{12}(m+1)(m+2)(m+3)(n+2) - \frac{1}{24}(m+1)(m+3)(m^2+4m+1)$ if m is odd. Therefore, if n and m are even, the decomposition of the representation of $W_{\mathbb{R}}$ to $V(n,0) \otimes V(m,0)$ contains $(\frac{n}{2}+1)(\frac{m}{2}+1)\phi_+ \oplus \frac{1}{2}(\frac{1}{12}(m+1)(m+2)(m+3)(n+2) - \frac{1}{24}m(m+2)^2(m+4) - (\frac{n}{2}+1)(\frac{m}{2}+1))(\phi_+ \oplus \phi_-)$ when $n+m \equiv 0 \ (4)$ and $(\frac{n}{2}+1)(\frac{m}{2}+1)\phi_- \oplus \frac{1}{2}(\frac{1}{12}(m+1)(m+2)(m+3)(n+2) - \frac{1}{24}m(m+2)^2(m+4) - (\frac{n}{2}+1)(\frac{m}{2}+1))(\phi_+ \oplus \phi_-)$ when $n+m \equiv 2 \ (4)$, and if n and m are odd, the decomposition of the representation of $W_{\mathbb{R}}$ to $V(n,0) \otimes V(m,0)$ contains $\frac{1}{2}(\frac{1}{12}(m+1)(m+2)(m+3)(n+2) - \frac{1}{24}(m+1)(m+3)(m^2+4m+1))(\phi_+ \oplus \phi_-)$.

Let $M(n+m-2i-j, j)$ be the multiplicity of the weight $(n+m-2i-j, j)$, $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$, in $V(n,0) \otimes V(m,0)$. The decomposition of $\hat{\mu} \circ \zeta$ for $\hat{\mu} : \mathrm{Sp}(4) \to \mathrm{GL}(V(n,0) \otimes V(m,0))$ is as follows.

$$
\begin{aligned}
&V(n,0) \otimes V(m,0) \\
&= \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)k+jl} \\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)k-jl} \\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)l+jk} \\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)l-jk} \\
&+ \bigoplus_{i<\frac{n+m}{2}} M(n+m-2i,0)\phi_{(n+m-2i)k} \\
&+ \bigoplus_{i<\frac{n+m}{2}} M(n+m-2i,0)\phi_{(n+m-2i)l}
\end{aligned}
$$

$$
+\begin{cases}
0 & n \neq m\ (2) \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 0\ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 0\ (4) \\
+(\frac{n}{2}+1)(\frac{m}{2}+1)\phi_+ & \\
+(\frac{1}{48}m(m+2)(m+4)(2n-m+2)(\phi_+ \oplus \phi_-) & \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 0\ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 2\ (4) \\
+(\frac{n}{2}+1)(\frac{m}{2}+1)\phi_- & \\
+(\frac{1}{48}m(m+2)(m+4)(2n-m+2)(\phi_+ \oplus \phi_-) & \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 1\ (2) \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & \\
+(\frac{1}{48}(m+1)(m+2)(m+3)(2n-m+2) & \\
\quad +\frac{1}{16}(m+1)(m+3))(\phi_+ \oplus \phi_-) &
\end{cases}
$$

Note that in this decomposition, ϕ_{ak+bl} may be equal to $\phi_0 = \phi_+ \oplus \phi_-$ for $ak+bl=0$. Using the multiplicities from Theorem 2.5 along with the earlier results of this section and the table of L- and ε-factors in Section 3.1, this decomposition provides the framework for calculating the archimedean factors of any representation of Sp(4).

We can also write the decomposition of a particular irreducible representation $V(n,m)$ into ϕ_p, ϕ_+, and ϕ_-. Using earlier calculations, the decomposition comes down to pairing weight spaces $W(a,b)$ and $W(-a,-b)$ for $(a,b) \neq (0,0)$ into two-dimensional representations of $W_{\mathbb{R}}$ and then separately considering $W(0,0)$ which will decompose into one-dimensional representations of $W_{\mathbb{R}}$.

Once again, we will use Corollary 2.4,

$$
\begin{aligned}
V(n,0) &= V(n,0)\otimes V(0,0) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 0, \\
V(n,1) &= V(n,0)\otimes V(1,0) - V(n+1,0)\otimes V(0,0) - V(n-1,0)\otimes V(0,0) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq 1, \\
V(n,m) &= V(n,0)\otimes V(m,0) + V(n,0)\otimes V(m-2,0) \\
&\qquad - V(n-1,0)\otimes V(m-1,0) - V(n+1,0)\otimes V(m-1,0) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geq m \geq 2.
\end{aligned}
$$

The decomposition of $W(0,0)$ in $V(n,m)$ can now be calculated using these results and the explicit description of the decomposition of $W(0,0)$ in $V(n,0)\otimes V(m,0)$ as it appears above.

Let $M(n+m-2i-j,j)$ be the multiplicity of the weight $(n+m-2i-j,j)$, $0 \leq i \leq \lfloor \frac{n+m}{2} \rfloor$ and $0 \leq j \leq \lfloor \frac{n+m}{2} \rfloor - i$, in $V(n,m)$. The decomposition of $\hat{\mu}\circ\zeta$ for $\hat{\mu} : \mathrm{Sp}(4) \to \mathrm{GL}(V(n,m))$ is as follows.

$$
\begin{aligned}
&V(n,m)\\
&= \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)k+jl}\\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)k-jl}\\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)l+jk}\\
&+ \bigoplus_{0<j<\frac{n+m}{2}-i} M(n+m-2i-j,j)\phi_{(n+m-2i-j)l-jk}\\
&+ \bigoplus_{i<\frac{n+m}{2}} M(n+m-2i,0)\phi_{(n+m-2i)k}\\
&+ \bigoplus_{i<\frac{n+m}{2}} M(n+m-2i,0)\phi_{(n+m-2i)l}
\end{aligned}
$$

$$
+\begin{cases}
0 & n \neq m \ (2) \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 0 \ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 0 \ (4) \\
+\frac{1}{4}(m+2)(n-m+2)\phi_+ & \\
+\frac{1}{4}m(n-m)\phi_- & \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 0 \ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 2 \ (4) \\
+\frac{1}{4}m(n-m)\phi_+ & \\
+\frac{1}{4}(m+2)(n-m+2)\phi_- & \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 1 \ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 0 \ (4) \\
+\frac{1}{4}(m+1)(n-m)\phi_+ & \\
+\frac{1}{4}(m+1)(n-m+2)\phi_- & \\
\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k+l)} & n \equiv m \equiv 1 \ (2), \\
+\bigoplus_{i<\frac{n+m}{2}} M(\frac{n+m}{2}-i, \frac{n+m}{2}-i)\phi_{(\frac{n+m}{2}-i)(k-l)} & n+m \equiv 2 \ (4) \\
+\frac{1}{4}(m+1)(n-m+2)\phi_+ & \\
+\frac{1}{4}(m+1)(n-m)\phi_- &
\end{cases}
$$

Chapter 4

Rank m symplectic Lie algebras

4.1 The case of $\mathfrak{sp}(2m, \mathbb{C})$

We will generalize the results of Chapter 1 for $\mathfrak{sp}(4, \mathbb{C})$ to representations of the Lie algebra

$$\mathfrak{sp}(2m, \mathbb{C}) = \{A \in \mathfrak{gl}(2m, \mathbb{C}) \mid A^t J + JA = 0\}.$$

Here $J = \left[\begin{smallmatrix} 0 & J_m \\ -J_m & 0 \end{smallmatrix}\right]$ and J_m is defined to be the $m \times m$ anti-diagonal matrix with ones along the anti-diagonal. Evidently, $\mathfrak{sp}(2m, \mathbb{C})$ is $(2m^2 + m)$-dimensional and has the following basis,

$$\{H_k\} = \{e_{kk} - e_{2m+1-k,2m+1-k} | k = 1, \ldots, m\},$$

$$\{X_\alpha\} = \{e_{ij} - e_{2m+1-j,2m+1-i}|$$
$$(i,j) = (1,2), \ldots, (1,m), (2,3), \ldots, (2,m), \ldots, (m-1,m)\}$$
$$\cup \{e_{i,2m+1-j} + e_{j,2m+1-i}|$$
$$(i,j) = (1,2), \ldots, (1,m), (2,3), \ldots, (2,m), \ldots, (m-1,m)\}$$
$$\cup \{e_{i,2m+1-i}|i = 1, \ldots, m\},$$

and

$$\{Y_\alpha\} = \{Y_\alpha = X_\alpha^t\}.$$

In this basis, the Cartan subalgebra is $\mathfrak{h} = \langle H_1, \ldots, H_m\rangle$, and for each root α,

$$\mathfrak{s}^\alpha = \text{span}\{X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Any weight $(x_1, x_2, \ldots, x_m)$ can be thought of as the eigenvalues associated to H_1 through H_m, respectively, for the corresponding weight vector. The weights in the dominant Weyl chamber are $\{(x_1, \ldots, x_m) \in \mathbb{Z}^m : x_1 \geq x_2 \geq \ldots \geq x_m \geq 0\}$. Let $V(x_1, \ldots, x_m)$ be the irreducible representation with highest weight $(x_1, \ldots, x_m)$.

The Weyl dimension formula, tailored to our situation, appears in [6], Section 7.6.3. It states that

$$\dim V(x_1, \ldots, x_m) = \frac{\prod_{\alpha>0} wt(X_\alpha) \cdot ((x_1, \ldots, x_m) + wt(\delta))}{\prod_{\alpha>0} wt(X_\alpha) \cdot wt(\delta)}$$

where $wt(X_\alpha)$ is the weight of X_α in the adjoint representation, $\cdot$ is the normal dot product, δ is half of the sum of the positive roots, and $wt(\delta)$ is the weight of

δ in the adjoint representation.

The standard representation of $\mathfrak{sp}(2m, \mathbb{C})$ is $V(1, 0, \ldots, 0)$. It has the standard basis $\{e_1, \ldots, e_{2m}\}$ and is isomporphic to its dual representation with corresponding basis $\{f_1, \ldots, f_{2m}\}$. These representations are isomorphic via $f_i \mapsto e_{2m+1-i}$ for $m+1 \leq i \leq 2m$ and $f_j \mapsto -e_{2m+1-j}$ for $1 \leq j \leq m$. The weights of $V(1, 0, \ldots, 0)$ are

$$\{(1, 0, \ldots, 0), \ldots, (0, \ldots, 1), (0, \ldots, 0, -1), \ldots, (-1, 0, \ldots, 0)\},$$

and e_1 is a highest weight vector.

It can be easily shown that $V(n, 0, \ldots, 0) = \mathrm{Sym}^n V(1, 0, \ldots, 0)$. First, there is a highest weight vector, $\mathrm{sym}(e_1 \otimes \ldots \otimes e_1)$, in $\mathrm{Sym}^n V(1, 0, \ldots, 0)$ with weight $(n, 0, \ldots, 0)$, and therefore $V(n, 0, \ldots, 0) \subset \mathrm{Sym}^n V(1, 0, \ldots, 0)$. Then using the Weyl dimension formula from above, $V(n, 0, \ldots, 0)$ has the same dimension as $\mathrm{Sym}^n V(1, 0, \ldots, 0)$ and thus $V(n, 0, \ldots, 0) = \mathrm{Sym}^n V(1, 0, \ldots, 0)$.

Proposition 4.1. *For $\mathfrak{sp}(2m, \mathbb{C})$ and $V = V(1, 0, \ldots, 0)$, the standard representation,*

$$\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V = (\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V) \oplus \bigoplus_{p=0}^{y} V(x+y-p, p, 0, \ldots, 0)$$

for integers $x \geq y \geq 1$.

Proof. Given $x \geq y$ and using the previously described basis, we define for all integers p such that $0 \leq p \leq y$ the following vector in $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V^*$,

$$\begin{aligned} v_p &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i (x-p+i, p-i, 0, \dots, 0) \times (0, \dots, 0, i, y-i) \\ &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mathrm{sym}(\underbrace{e_1 \otimes \dots \otimes e_1}_{x-p+i} \otimes \underbrace{e_2 \otimes \dots \otimes e_2}_{p-i}) \\ &\qquad \otimes \mathrm{sym}(\underbrace{f_{2m-1} \otimes \dots \otimes f_{2m-1}}_{i} \otimes \underbrace{f_{2m} \otimes \dots \otimes f_{2m}}_{y-i}). \end{aligned}$$

This vector is in the kernel of the map ρ^* defined in Section 2.1 because

$$(x-p+i, p-i, 0, \dots, 0) \times (0, \dots, 0, i, y-i) \mapsto 0 + \dots + 0 = 0.$$

Also, this vector is a highest weight vector with weight

$$\begin{aligned} &(x-p+i)(1,0,\dots,0) + (p-i)(0,1,0,\dots,0) \\ &\qquad + i(0,1,0,\dots,0) + (y-i)(1,0,\dots,0) = (x+y-p, p, 0, \dots, 0). \end{aligned}$$

To see v_p is a highest weight vector, it is enough to show that it is in the kernel of X_α for any α.

First, the only relevant calculations are $X_\alpha.e_1$, $X_\alpha.e_2$, $X_\alpha.f_{2m}$, and $X_\alpha.f_{2m-1}$. These will all be equal to zero except when $X_\alpha = e_{12} - e_{2m-1,2m}$. Therefore, we only need to show v_p is in the kernel of $X_\alpha = e_{12} - e_{2m-1,2m}$. Call this root X_{12}.

$X_{12}.e_1 = X_{12}.f_{2m} = 0$, $X_{12}.e_2 = e_1$, and $X_{12}.f_{2m-1} = f_{2m}$. By definition,

$$
\begin{aligned}
X_{12}.&(x-p+i, p-i, 0, \dots, 0) \times (0, \dots, 0, i, y-i) \\
&= X_{12}.\mathrm{sym}(\underbrace{e_1 \otimes \dots \otimes e_1}_{x-p+i} \otimes \underbrace{e_2 \otimes \dots \otimes e_2}_{p-i}) \\
&\qquad\qquad \otimes \mathrm{sym}(\underbrace{f_{2m-1} \otimes \dots \otimes f_{2m-1}}_{i} \otimes \underbrace{f_{2m} \otimes \dots \otimes f_{2m}}_{y-i}).
\end{aligned}
$$

This becomes

$$
\begin{aligned}
&(x-p+i)\mathrm{sym}(X_{12}.e_1 \otimes e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_2) \\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \dots \otimes f_{2m-1} \otimes f_{2m} \otimes \dots \otimes f_{2m}) \\
&+ (p-i)\mathrm{sym}(e_1 \otimes \dots \otimes e_1 \otimes X_{12}.e_2 \otimes e_2 \otimes \dots \otimes e_2) \\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \dots \otimes f_{2m-1} \otimes f_{2m} \otimes \dots \otimes f_{2m}) \\
&+ (i)\mathrm{sym}(e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_2) \\
&\qquad \otimes \mathrm{sym}(X_{12}.f_{2m-1} \otimes f_{2m-1} \otimes \dots \otimes f_{2m-1} \otimes f_{2m} \otimes \dots \otimes f_{2m}) \\
&+ (y-i)\mathrm{sym}(e_1 \otimes \dots \otimes e_1 \otimes e_2 \otimes \dots \otimes e_2) \\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \dots \otimes f_{2m-1} \otimes X_{12}.f_{2m} \otimes f_{2m} \otimes \dots \otimes f_{2m}).
\end{aligned}
$$

This is equal to $(x-p+i)(0) + (p-i)(x-p+i+1, p-i-1, 0, \dots, 0) \times (0, \dots, 0, i, y-i) + (i)(x-p+i, p-i, 0, \dots, 0) \times (0, \dots, 0, i-1, y-i+1) + (y-i)(0)$ (with the understanding that when $i = p$ there is no second term and when $i = 0$ there is no third term here). From here $X_{12}.v_p = 0$ is a straightforward calculation.

For each of these highest weight vectors, v_p, with weight $(x+y-p, p, 0 \dots, 0)$ and in $\ker(\rho^*)$, there is an irreducible representation $V(x+y-p, p, 0, \dots, 0)$ contained in the kernel. Since all of the weights $\{(x+y-p, p, 0 \ldots, 0) : 0 \le p \le y\}$, are distinct, $\bigoplus_{p=0}^{y} V(x+y-p, p, 0, \ldots, 0) \subset \ker(\rho^*)$.

It follows from semisimplicity and the surjectivity of ρ^* that

$$\begin{aligned}
(\mathrm{Sym}^{x-1}V \otimes \mathrm{Sym}^{y-1}V^*) \oplus \bigoplus_{p=0}^{y} V(x+y-p, p, 0, \ldots, 0) \\
\subset (\mathrm{Sym}^{x-1}V \otimes \mathrm{Sym}^{y-1}V^*) \oplus \ker(\rho^*) \\
= \mathrm{Sym}^{x}V \otimes \mathrm{Sym}^{y}V^*
\end{aligned}$$

for $x \ge y \ge 1$. The Weyl dimension formula shows that this inclusion is actually an equality. Note that V^* can be replaced by V since this representation is self-dual. □

Note that all of the highest weight vectors in $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V$, for $V = V(1, 0, \ldots, 0)$, can be determined using the proof of Proposition 4.1, the map ρ from Section 2.1, and the isomorphism between the standard representation and its dual.

Corollary 4.2. *For integers $x \ge y = 1$,*

$$\begin{aligned}
&V(x, 0, \ldots, 0) \otimes V(1, 0, \ldots, 0) \\
&= V(x+1, 0, \ldots, 0) \oplus V(x, 1, 0, \ldots, 0) \oplus V(x-1, 0, \ldots, 0).
\end{aligned}$$

For $x \ge y \ge 2$,

$$\begin{aligned}
(V(x, 0, \ldots, 0) \otimes V(y, 0, \ldots, 0)) \oplus (V(x, 0, \ldots, 0) \otimes V(y-2, 0, \ldots, 0)) \\
= (V(x+1, 0, \ldots, 0) \otimes V(y-1, 0, \ldots, 0)) \oplus V(x, y, 0, \ldots, 0) \\
\oplus (V(x-1, 0, \ldots, 0) \otimes V(y-1, 0, \ldots, 0)).
\end{aligned}$$

Proof. Recall $\mathrm{Sym}^n V(1,0,\ldots,0) = V(n,0,\ldots,0)$. The first assertion is the special case of Proposition 4.1 where $y=1$. Using Proposition 4.1, when $x \geq y \geq 2$,

$$\begin{aligned} &V(x,0,\ldots,0) \otimes V(y,0,\ldots,0) \\ &\qquad = (V(x-1,0,\ldots,0) \otimes V(y-1,0,\ldots,0)) \oplus \bigoplus_{p=0}^{y} V(x+y-p,p,0,\ldots,0) \end{aligned}$$

and

$$\begin{aligned} &V(x+1,0,\ldots,0) \otimes V(y-1,0,\ldots,0) \\ &\qquad = (V(x,0,\ldots,0) \otimes V(y-2,0,\ldots,0)) \oplus \bigoplus_{p=0}^{y-1} V(x+y-p,p,0,\ldots,0). \end{aligned}$$

Combining these equations yields the assertion. □

In the Grothendieck group of all representations of $\mathfrak{sp}(2m,\mathbb{C})$, setting $V = V(1,0,\ldots,0)$, we get

$$\begin{aligned} V(x,0,\ldots,0) &= \mathrm{Sym}^x V & x \geq 0 \\ V(x,1,0,\ldots,0) &= \mathrm{Sym}^x V \otimes V - \mathrm{Sym}^{x+1} V - \mathrm{Sym}^{x-1} V & x \geq 1 \\ V(x,y,0,\ldots,0) &= \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V \otimes \mathrm{Sym}^{y-2} V & x \geq y \geq 2 \\ &\quad - \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V - \mathrm{Sym}^{x+1} V \otimes \mathrm{Sym}^{y-1} V. \end{aligned}$$

4.2 The Littlewood-Richardson rule for $\mathrm{Sp}(2m)$

This idea of using the standard representation as a building block for determining every irreducible representation can then be expanded to more complicated highest weights, but more machinery is needed. In [10], Littelmann provides a generalization of the Littlewood-Richardson rule in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. The main result from [10] is the following theorem.

Theorem 4.3. *The decomposition of the tensor product $V_\lambda \otimes V_\mu$ into irreducible G-modules is given by*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathrm{T}} V_{\lambda+v(\mathrm{T})}$$

where T *runs over all G-standard Young tableaux of shape $p(\mu)$ that are λ-dominant.*

Let $G = \mathrm{Sp}(2m)$. We will now give a description, tailored to our situation, of the $\mathrm{Sp}(2m)$-standard Young tableaux of shape $p(\mu)$ that are λ-dominant. We will only need to consider the case where $\mu = (n, 0, \ldots, 0)$.

The $\mathrm{Sp}(2m)$-standard Young tableaux of shape $p(n, 0, \ldots, 0)$ are all of the Young diagrams consisting of a single nondecreasing column of length n containing the integers 1 to $2m$.

Define $v(\mathrm{T}) := (c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m))\epsilon_1 + (c_{\mathrm{T}}(2) - c_{\mathrm{T}}(2m-1))\epsilon_2 + \ldots + (c_{\mathrm{T}}(m) - c_{\mathrm{T}}(m+1))\epsilon_m$, where $c_{\mathrm{T}}(i)$ is equal to the number of times the number i appears in the tableau T.

Let $\mathrm{T}(l)$ be the tableau created from T by removing rows $l+1$ to n, counting from bottom to top. Then an $\mathrm{Sp}(2m)$-standard Young tableau T of shape $p(n, 0, \ldots, 0)$ is λ-dominant if all of the weights $\lambda + v(\mathrm{T}(l))$ are dominant weights for $1 \leq l \leq n$.

We can now present the main theorem of this note.

Theorem 4.4. *Any irreducible representation of $\mathfrak{sp}(4, \mathbb{C})$, $V(x_1, \ldots, x_k, 0, \ldots, 0)$,* *can be written as an integral combination in the form*

$$\sum_{i=1}^{n} c_i V(y_1^{(i)}, 0, \ldots, 0) \otimes \ldots \otimes V(y_k^{(i)}, 0, \ldots, 0) \tag{4.1}$$

for some n, c_i, and $y_j^{(i)}$.

Note that $V(n, 0, \ldots, 0) = \mathrm{Sym}^n V(1, 0, \ldots, 0)$, and the tensor products in the formal sum are products of varying symmetric powers of the standard representation.

Proof. We will use induction on k. The case where $k = 1$ is trivial, and the case where $k = 2$ is a consequence of Corollary 4.2.

Assume the statement of the theorem is true for k. Now, we will want to show $V(x_1, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)}, 0, \ldots, 0) \otimes \ldots \otimes V(y_{k+1}^{(i)}, 0, \ldots, 0)$ for some n, c_i, and $y_j^{(i)}$. We will prove this assertion by induction on the size of x_{k+1}.

When $x_{k+1} = 0$, $V(x_1, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)}, 0, \ldots, 0) \otimes \ldots \otimes V(y_{k+1}^{(i)}, 0, \ldots, 0)$, with $y_{k+1}^{(i)} = 0$ for every i, using the inductive hypothesis for the induction on k.

Assume true for $x_{k+1} \leq z - 1$. Now we want to show $V(x_1, \ldots, x_k, z, 0, \ldots, 0)$, for any x_i and z such that $x_1 \geq \ldots \geq x_k \geq z$, can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)}, 0, \ldots, 0) \otimes \ldots \otimes V(y_{k+1}^{(i)}, 0, \ldots, 0)$ for some n, c_i, and $y_j^{(i)}$. Consider the decomposition of $V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0)$ using Theorem 4.3.

Consider the standard Young tableaux, T, of shape $p(z, 0, \ldots, 0)$, which are also $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant. These are the nondecreasing columns of length z with entries taken from the set of integers between 1 and $k + 1$ and integers between $2m+1-k$ and $2m$ such that the following inequalities are satisfied for $1 \leq i \leq k-1$.

$$x_i - c_{\mathrm{T}}(2m+1-i) \geq x_{i+1} \tag{4.2}$$

$$x_i - c_{\mathrm{T}}(2m+1-i) \geq x_{i+1} + c_{\mathrm{T}}(i+1) - c_{\mathrm{T}}(2m-i) \tag{4.3}$$

$$x_k - c_{\mathrm{T}}(2m+1-k) \geq c_{\mathrm{T}}(k+1) \tag{4.4}$$

In the decomposition of $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$ using Theorem 4.3, each irreducible representation has a highest weight

$$(x_1 + c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m), \ldots, x_k + c_{\mathrm{T}}(k) - c_{\mathrm{T}}(2m+1-k), c_{\mathrm{T}}(k+1), 0, \ldots, 0)$$

for some standard Young tableaux, T, with shape $p(z,0,\ldots,0)$ and which is also $(x_1,\ldots,x_k,0,\ldots,0)$-dominant. All of these highest weights have

$$0 \leq c_{\mathrm{T}}(k+1) \leq z-1$$

except in the case where T, a column of length z, contains only entries equal to $k+1$ with $c_{\mathrm{T}}(k+1) = z$. In this case, the highest weight is $(x_1, x_2, \ldots, x_k, z, 0, \ldots, 0)$. By induction, every other irreducible representation in the decomposition, except for $V(x_1,\ldots,x_k,z,0,\ldots,0)$, can be written as a some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$. $V(x_1,\ldots,x_k,0,\ldots,0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_k^{(i)},0,\ldots,0)$ by the inductive hypothesis for the induction on k, so that $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$ is equivalent to $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_k^{(i)},0,\ldots,0) \otimes V(z,0\ldots,0)$.

By isolating the representation $V(x_1, \ldots, x_k, z, 0, \ldots, 0)$ in the decomposition of $V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0)$, $V(x_1, \ldots, x_k, z, 0, \ldots, 0)$ can now be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)}, 0, \ldots, 0) \otimes \ldots \otimes V(y_{k+1}^{(i)}, 0, \ldots, 0)$. This completes the induction on z and thus the induction on k.

□

This proof provides a recursive algorithm for finding the formal combination as in (4.1) for any irreducible representation. For example, the first step in determining (4.1) for $V(x_1, \ldots, x_k, 1, 0, \ldots, 0)$ is the following identification:

$$\begin{aligned} V(x_1, \ldots, x_k, 1, 0, \ldots, 0) = V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(1, 0, \ldots, 0) \\ -\Big(\bigoplus_{\substack{x_{i-1} \neq x_i \\ i=1,\ldots,k}} V(x_1, \ldots, x_i + 1, \ldots, x_k, 0, \ldots, 0)\Big) \\ -\Big(\bigoplus_{\substack{x_i \neq x_{i+1} \\ i=1,\ldots,k}} V(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0)\Big). \end{aligned}$$

4.3 A refinement of the recursive algorithm

For $V(x_1, \ldots, x_{k+1}, 0, \ldots, 0)$, assume $z = x_{k+1} \geq 2k$ and all of the representations of the form $V(x_1, \ldots, x_k, 0, \ldots, 0)$ have known integral combinations in the form of (4.1). Define the following algorithm.

For $x_1 \neq x_2$, define the representation

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m) \\ &:= V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0) \\ &\quad - V(x_1 - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z - 1, 0, \ldots, 0). \end{aligned}$$

If $x_1 = x_2$, *define the representation*

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m) \\ &\quad := V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0). \end{aligned}$$

For $x_i \neq x_{i+1}$ *with* $2 \leq i \leq k-1$ *or* $x_i \geq x_{i+1}$ *with* $i = k$, *define the representation*

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+1-i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+2-i) \\ &\qquad - F(V(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0), 2m+2-i). \end{aligned}$$

For $x_i = x_{i+1}$, $2 \leq i \leq k-1$, *define the representation*

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+1-i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+2-i). \end{aligned}$$

For $x_i \neq x_{i-1}$, $2 \leq i \leq k$, *define the representation*

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i+1) \\ &\qquad - F(V(x_1, \ldots, x_i + 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0), i+1). \end{aligned}$$

For $x_i = x_{i-1}$, $2 \leq i \leq k$, *define the representation*

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i+1). \end{aligned}$$

Define the representation

$$
\begin{aligned}
&F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),1)\\
&\quad := F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2)\\
&\qquad - F(V(x_1+1,x_2,\ldots,x_k,0,\ldots,0)\otimes V(z-1,0,\ldots,0),2).
\end{aligned}
$$

Then,

$$V(x_1,\ldots,x_k,z,0,\ldots,0) = F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),1).$$

This algorithm produces an integral combination equal to the representation $V(x_1,\ldots,x_k,z,0,\ldots,0)$ of representations of the form $V(x'_1,\ldots,x'_k,0,\ldots,0)\otimes V(z',0,\ldots,0)$. Substituting in the integral combinations in the form of (1) for the representations $V(x'_1,\ldots,x'_k,0,\ldots,0)$ yields the integral combination in the form of (1) for $V(x_1,\ldots,x_k,z,0,\ldots,0)$. The following is an explanation of how the algorithm works.

Recall that

$$V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0) = \bigoplus_{\mathrm{T}} V((x_1,\ldots,x_k,0,\ldots,0)+v(\mathrm{T}))$$

for all standard Young tableaux, T, of shape $p(z,0,\ldots,0)$ and which are also $(x_1,\ldots,x_k,0,\ldots,0)$-dominant, which means all single nondecreasing columns, T, of length z containing integers from the set of integers between 1 and $k+1$ and integers between $2m+1-k$ and $2m$ and satisfying conditions (2), (3), and (4).

If $x_i \neq x_{i+1}$, define a map from the standard Young tableaux, $\tilde{\mathrm{T}}$, of shape $p(z-1, 0, \ldots, 0)$ that are $(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain integers from the set $\{2m+2-i, \ldots, 2m\}$ to the standard Young tableaux, T, of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain integers from the set $\{2m+2-i, \ldots, 2m\}$ by sending $\tilde{\mathrm{T}}$ to the tableau formed by adding a $2m+1-i$ to the bottom of the column.

This map is a bijection between all of the $\tilde{\mathrm{T}}$ and all of the T containing a $2m+1-i$, taking into account the conditions (2), (3), and (4). The map preserves the highest weights of the representations corresponding to these tableaux in the respective decompositions $V(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0)$ and $V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0)$ using Littelmann's theorem, Theorem 4.3. For $\tilde{\mathrm{T}} \mapsto \mathrm{T}$, $c_{\mathrm{T}}(j) = c_{\tilde{\mathrm{T}}}(j)$ for all $j \neq 2m+1-i$, and $c_{\mathrm{T}}(2m+1-i) = c_{\tilde{\mathrm{T}}}(2m+1-i) + 1$. Therefore,

$$\begin{aligned}
&V(x_1 + c_{\tilde{\mathrm{T}}}(1) - c_{\tilde{\mathrm{T}}}(2m), \ldots, x_i - 1 + c_{\tilde{\mathrm{T}}}(i) - c_{\tilde{\mathrm{T}}}(2m+1-i), \ldots, \\
&\qquad x_k + c_{\tilde{\mathrm{T}}}(k) - c_{\tilde{\mathrm{T}}}(2m+1-k), c_{\tilde{\mathrm{T}}}(k+1), 0, \ldots, 0) \\
&= V(x_1 + c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m), \ldots, x_i + c_{\mathrm{T}}(i) - c_{\mathrm{T}}(2m+1-i), \ldots, \\
&\qquad x_k + c_{\mathrm{T}}(k) - c_{\mathrm{T}}(2m+1-k), c_{\mathrm{T}}(k+1), 0, \ldots, 0).
\end{aligned}$$

For $x_i \neq x_{i+1}$, when $i = 1$, the representation

$$\begin{aligned}
&F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m) \\
&:= V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0) \\
&\qquad - V(x_1 - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0)
\end{aligned}$$

and when $2 \leq i \leq k-1$, the representation

$$\begin{aligned}
&F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+1-i)\\
&\quad := F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+2-i)\\
&\quad\ - F(V(x_1,\ldots,x_i-1,\ldots,x_k,0,\ldots,0)\otimes V(z-1,0,\ldots,0),2m+2-i)
\end{aligned}$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1,\ldots,x_k,0,\ldots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and do not contain any integer from the set $\{2m+1-i,\ldots,2m\}$.

For $x_i = x_{i+1}$, when $i = 1$, the representation

$$\begin{aligned}
&F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m)\\
&\quad := V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0)
\end{aligned}$$

and when $2 \leq i \leq k-1$, the representation

$$\begin{aligned}
&F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+1-i)\\
&\quad := F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+2-i)
\end{aligned}$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1,\ldots,x_k,0,\ldots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and do not contain any integer from the set $\{2m+1-i,\ldots,2m\}$.

For $i = k$, it is only important that $x_k \geq z$, which is true for any highest weight. The representation

$$\begin{aligned}
&F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+1-k)\\
&\quad := F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),2m+2-k)\\
&\qquad - F(V(x_1,\ldots,x_k-1,0,\ldots,0)\otimes V(z-1,0,\ldots,0),2m+2-k)
\end{aligned}$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1,\ldots,x_k,0,\ldots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and do not contain any integer from the set $\{2m+1-k,\ldots,2m\}$.

The standard Young tableaux, T, of shape $p(z,0,\ldots,0)$ and which are also $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and do not contain any integer from the set $\{2m+1-k,\ldots,2m\}$ will only contain integers from the set $\{1,\ldots,k+1\}$. Note that if T does not contain an integer i, this is the same as saying $c_{\mathrm{T}}(i)=0$.

For $x_i \neq x_{i-1}$, define a map from the standard Young tableaux, $\tilde{\mathrm{T}}$, of shape $p(z-1,0,\ldots,0)$ that are $(x_1,\ldots,x_i+1,\ldots,x_k,0,\ldots,0)$-dominant and only contain integers from the set $\{1,\ldots,i,k+1\}$ to the standard Young tableaux, T, of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and only contain integers from the set $\{1,\ldots,i,k+1\}$ by sending $\tilde{\mathrm{T}}$ to the tableau formed by adding an i to the column.

This map is a bijection between all of the $\tilde{\mathrm{T}}$ and all of the T containing an i, taking into account the conditions (2), (3), and (4). The map preserves the highest weights of the representations corresponding to these tableaux in the respective decompositions $V(x_1,\ldots,x_i+1,\ldots,x_k,0,\ldots,0)\otimes V(z-1,0,\ldots,0)$ and $V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0)$ using Littelmann's theorem, Theorem

4.3. For $\tilde{\mathrm{T}} \mapsto \mathrm{T}$, $c_{\mathrm{T}}(j) = c_{\tilde{\mathrm{T}}}(j)$ for all $j \neq i$, and $c_{\mathrm{T}}(i) = c_{\tilde{\mathrm{T}}}(i) + 1$. Therefore,

$$\begin{aligned}
&V(x_1 + c_{\tilde{\mathrm{T}}}(1) - c_{\tilde{\mathrm{T}}}(2m), \ldots, x_i + 1 + c_{\tilde{\mathrm{T}}}(i) - c_{\tilde{\mathrm{T}}}(2m+1-i), \ldots, \\
&\quad x_k + c_{\tilde{\mathrm{T}}}(k) - c_{\tilde{\mathrm{T}}}(2m+1-k), c_{\tilde{\mathrm{T}}}(k+1), 0, \ldots, 0) \\
&= V(x_1 + c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m), \ldots, x_i + c_{\mathrm{T}}(i) - c_{\mathrm{T}}(2m+1-i), \ldots, \\
&\quad x_k + c_{\mathrm{T}}(k) - c_{\mathrm{T}}(2m+1-k), c_{\mathrm{T}}(k+1), 0, \ldots, 0).
\end{aligned}$$

For $x_i \neq x_{i-1}$, when $i = k$, the representation

$$\begin{aligned}
&F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), k) \\
&:= V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0) \\
&\quad - V(x_1, \ldots, x_k + 1, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0)
\end{aligned}$$

and when $2 \leq i \leq k-1$, the representation

$$\begin{aligned}
&F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i) \\
&:= F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i+1) \\
&\quad - F(V(x_1, \ldots, x_i + 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0), i+1)
\end{aligned}$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and only contain integers from the set $\{1, \ldots, i-1, k+1\}$.

For $x_i = x_{i-1}$, when $i = k$, the representation

$$F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), k)$$
$$:= F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m + 1 - k)$$

and when $2 \leq i \leq k - 1$, the representation

$$F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i)$$
$$:= F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), i + 1)$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and only contain integers from the set $\{1, \ldots, i - 1, k + 1\}$.

For $i = 1$, there are no restrictions on the number of times 1 appears in a tableau (other than the size of the tableau), the representation

$$F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 1)$$
$$:= F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2)$$
$$- F(V(x_1 + 1, x_2, \ldots, x_k, 0, \ldots, 0) \otimes V(z - 1, 0, \ldots, 0), 2)$$

is equal to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and only contain integers from the set $\{k+1\}$. The only tableau satisfying these conditions is the single column containing only $k + 1$s. This tableau corresponds to the representation

$V(x_1, \ldots, x_k, z, 0, \ldots, 0)$. Therefore,

$$V(x_1, \ldots, x_k, z, 0, \ldots, 0)$$
$$= F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 1),$$

which is an integral combination of representations

$$V(y_1, \ldots, y_k, 0, \ldots, 0) \otimes V(z - i, 0, \ldots, 0)$$

for some y_j and i. Substituting in the integral combinations for all of the $V(y_1, \ldots, y_k, 0, \ldots, 0)$ yields the integral combination of $V(x_1, \ldots, x_k, z, 0, \ldots, 0)$ with $z = x_{k+1}$ in the form of (4.1).

This algorithm can also be used when $z < 2k$. It can be applied until the size of z is exhausted, thus simplifying the problem of determining the integral combination to a reduced number of tableaux. If there are some equal terms, $x_i = x_{i+1}$, the algorithm may be completed for some $z < 2k$.

This algorithm also produces the following formula.

Proposition 4.5. *For any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ with highest weight $(x_1, \ldots, x_k, z, 0, \ldots, 0)$, such that $x_i \geq x_{i+1} + 2$ when $1 \leq i \leq k-1$ and $z \geq 2k$,*

$$V(x_1, \ldots, x_k, z, 0, \ldots, 0)$$
$$= \sum_{\substack{i_1, \ldots, i_k \in \{0,1\} \\ j_1, \ldots, j_k \in \{0,1\}}} (-1)^{i+j} \big(V(x_1 - i_1 + j_1, x_2 - i_2 + j_2, \ldots, x_k - i_k + j_k, 0, \ldots, 0)$$
$$\otimes V(z - i - j, 0, \ldots, 0) \big)$$

for $i = i_1 + \dots i_k$ *and* $j = j_1 + \dots j_k$.

4.4 Examples

From earlier results, for any irreducible representation of $\mathfrak{sp}(4, \mathbb{C})$ and $V = V(1, 0)$, its formal combination is determined by

$$\begin{aligned} V(x,0) &= \mathrm{Sym}^x V & x \geq 0 \\ V(x,1) &= \mathrm{Sym}^x V \otimes V - \mathrm{Sym}^{x+1} V - \mathrm{Sym}^{x-1} V & x \geq 1 \\ V(x,y) &= \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V \otimes \mathrm{Sym}^{y-2} V & x \geq y \geq 2 \\ &\quad - \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V - \mathrm{Sym}^{x+1} V \otimes \mathrm{Sym}^{y-1} V. \end{aligned}$$

This example can also be found using the results of the previous two sections. To apply the refinement of the recursive algorithm to the case of $\mathfrak{sp}(4, \mathbb{C})$ and some $V(x, y)$ such that $x \geq y \geq 2$, we do the following.

$$F(V(x,0) \otimes V(y,0), 4) = V(x,0) \otimes V(y,0) - V(x-1,0) \otimes V(y-1,0).$$

$$F(V(x,0) \otimes V(y,0), 1) = F(V(x,0) \otimes V(y,0), 4) - F(V(x+1,0) \otimes V(y-1,0), 4)$$

and

$$F(V(x+1,0) \otimes V(y-1,0), 4) = V(x+1,0) \otimes V(y-1,0) - V(x,0) \otimes V(y-2,0).$$

Therefore,

$$\begin{aligned}
&F(V(x,0)\otimes V(y,0),1)\\
&=V(x,0)\otimes V(y,0)-V(x-1,0)\otimes V(y-1,0)\\
&\qquad-(V(x+1,0)\otimes V(y-1,0)-V(x,0)\otimes V(y-2,0))\\
&=V(x,0)\otimes V(y,0)-V(x-1,0)\otimes V(y-1,0)\\
&\qquad-V(x+1,0)\otimes V(y-1,0)+V(x,0)\otimes V(y-2,0),
\end{aligned}$$

and

$$\begin{aligned}
&V(x,y)\\
&=F(V(x,0)\otimes V(y,0),1)\\
&=V(x,0)\otimes V(y,0)-V(x-1,0)\otimes V(y-1,0)\\
&\qquad-V(x+1,0)\otimes V(y-1,0)+V(x,0)\otimes V(y-2,0).
\end{aligned}$$

Equivalently,

$$V(x,y)=\sum_{i,j\in\{0,1\}}(-1)^{i+j}V(x-i+j,0)\otimes V(y-i-j,0).$$

For any irreducible representation of $\mathfrak{sp}(6,\mathbb{C})$ with highest weight $(x,y,0)$, its formal combination is determined similarly as above. For $V(x,y,z)$ such that $x\geq y+2$ and $z\geq 4$, the refinement to the recursive algorithm produces the following output,

$$
\begin{aligned}
&V(x,y,z) = \\
&\quad V(x,y,0)\otimes V(z,0,0) && - V(x-1,y,0)\otimes V(z-1,0,0)\\
&- V(x,y-1,0)\otimes V(z-1,0,0) && + V(x-1,y-1,0)\otimes V(z-2,0,0)\\
&- V(x,y+1,0)\otimes V(z-1,0,0) && + V(x-1,y+1,0)\otimes V(z-2,0,0)\\
&+ V(x,y,0)\otimes V(z-2,0,0) && - V(x-1,y,0)\otimes V(z-3,0,0)\\
&- V(x+1,y,0)\otimes V(z-1,0,0) && + V(x,y,0)\otimes V(z-2,0,0)\\
&+ V(x+1,y-1,0)\otimes V(z-2,0,0) && - V(x,y-1,0)\otimes V(z-3,0,0)\\
&+ V(x+1,y+1,0)\otimes V(z-2,0,0) && - V(x,y+1,0)\otimes V(z-3,0,0)\\
&- V(x+1,y,0)\otimes V(z-3,0,0) && + V(x,y,0)\otimes V(z-4,0,0).
\end{aligned}
$$

This is equivalent to

$$
V(x,y,z) = \sum_{\substack{i_1,i_2\in\{0,1\}\\ j_1,j_2\in\{0,1\}}} (-1)^{i+j} V(x-i_1+j_1, y-i_2+j_2, 0)\otimes V(z-i-j,0,0)
$$

where $i = i_1 + i_2$ and $j = j_1 + j_2$.

Substituting in for the irreducible representations with highest weights of the form $(x', y', 0)$ and simplifying, this becomes

$$
\begin{aligned}
&V(x,y,z)\\
&= \sum_{\substack{l_1,l_2,l_3\in\{0,\pm1,\pm2\}\\ \{|l_1|,|l_2|,|l_3|\}=\{0,1,2\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & 2 \\ |l_1| & |l_2| & |l_3| \end{pmatrix} V(x+l_1,0,0)\otimes V(y+l_2-1,0,0)\\
&\qquad\qquad\otimes V(z+l_3-2,0,0).
\end{aligned}
$$

Note that all of the coefficients in this sum are ± 1, but this is not always the case for any highest weight. For example, in $\mathfrak{sp}(6,\mathbb{C})$,

$$\begin{aligned} V(1,1,1) = {} & V(1,0,0) \otimes V(1,0,0) \otimes V(1,0,0) \\ & - 2V(2,0,0) \otimes V(1,0,0) + V(3,0,0) - V(1,0,0). \end{aligned}$$

4.5 A general formula

Now, we will expand the formula explicitly calculated in Section 4.4 for $V(x,y,z)$, such that $x \geq y+2$ and $z \geq 4$, to a general case for representations in $\mathfrak{sp}(2m,\mathbb{C})$ with highest weights of sufficient size.

Theorem 4.6. *For any irreducible representation of $\mathfrak{sp}(4,\mathbb{C})$, which is denoted by $V(x_1,\ldots,x_k,0,\ldots,0)$, such that $x_i \geq x_{i+1} + 2(k-1-i)$ when $1 \leq i \leq k-1$ and $x_k \geq 2k-2$,*

$$V(x_1,\ldots,x_k,0,\ldots,0) = \sum_{\substack{l_1,\ldots,l_k \in \{0,\pm 1,\ldots,\pm(k-1)\} \\ \{|l_1|,\ldots,|l_k|\} = \{0,1,\ldots,k-1\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \bigotimes_{n=1}^{k} V(x_n + l_n - n + 1, 0, \ldots, 0).$$

Proof. We will argue by induction on k. The case when $k=1$ is trivial. When

$k = 2$ and $x_2 \geq 2$,

$$\begin{aligned}
&V(x_1, x_2, 0, \dots, 0) \\
&= \sum_{i,j \in \{0,1\}} (-1)^{i+j} V(x_1 - i + j, 0, \dots, 0) \otimes V(x_2 - i - j, 0, \dots, 0) \\
&= \sum_{\substack{l_1, l_2, \in \{0, \pm 1\} \\ \{|l_1|, |l_2|\} = \{0,1\}}} \operatorname{sgn} \begin{pmatrix} 0 & 1 \\ \\ |l_1| & |l_2| \end{pmatrix} V(x_1 + l_1, 0, \dots, 0) \otimes V(x_2 + l_2 - 1, 0, \dots, 0) \\
&= V(x_1, 0) \otimes V(x_2, 0) - V(x_1 - 1, 0) \otimes V(x_2 - 1, 0) \\
&\qquad - V(x_1 + 1, 0) \otimes V(x_2 - 1, 0) + V(x_1, 0) \otimes V(x_2 - 2, 0).
\end{aligned}$$

Assume the statement of the theorem for k. Let $x_{k+1} = z$, we want to show

$$\begin{aligned}
&V(x_1, \dots, x_k, z, 0, \dots, 0) = \\
&\sum_{\substack{l'_1, \dots, l'_{k+1} \in \{0, \pm 1, \dots, \pm(k)\} \\ \{|l'_1|, \dots, |l'_{k+1}|\} = \{0, 1, \dots, k\}}} \operatorname{sgn} \begin{pmatrix} 0 & 1 & \dots & k \\ \\ |l'_1| & |l'_2| & \dots & |l'_{k+1}| \end{pmatrix} \bigotimes_{n=1}^{k+1} V(x_n + l'_n - n + 1, 0, \dots, 0)
\end{aligned}$$

for $x_i \geq x_{i+1} + 2(k - i)$ when $1 \leq i \leq k$ and $z \geq 2k$. Call this sum S'. The tensor products in this sum are indexed by a k-tuple, $(l'_1, \dots, l'_{k+1})$.

From Proposition 4.5,

$$\begin{aligned}
&V(x_1, \dots, x_k, z, 0, \dots, 0) \\
&= \sum_{\substack{i_1, \dots, i_k \in \{0,1\} \\ j_1, \dots, j_k \in \{0,1\}}} (-1)^{i+j} \big(V(x_1 - i_1 + j_1, x_2 - i_2 + j_2, \dots, x_k - i_k + j_k, 0, \dots, 0) \\
&\qquad\qquad\qquad\qquad \otimes V(z - i - j, 0, \dots, 0) \big).
\end{aligned}$$

Applying the inductive hypothesis to $V(x_1 - i_1 + j_1, \dots, x_k - i_k + j_k, 0, \dots, 0)$

yields the following,

$$
\begin{aligned}
&V(x_1,\ldots,x_k,z,0,\ldots,0)\\
&=\sum_{\substack{i_1,\ldots,i_k\in\{0,1\}\\ j_1,\ldots,j_k\in\{0,1\}}}\ \sum_{\substack{l_1,\ldots,l_k\in\{0,\pm1,\ldots,\pm(k-1)\}\\ \{|l_1|,\ldots,|l_k|\}=\{0,1,\ldots,k-1\}}}(-1)^{i+j}\operatorname{sgn}\begin{pmatrix}0 & 1 & \ldots & k-1\\ |l_1| & |l_2| & \ldots & |l_k|\end{pmatrix}\\
&\qquad\Big(\bigotimes_{n=1}^{k}V(x_n-i_n+j_n+l_n-(n-1),0,\ldots,0)\Big)\otimes V(z-i-j,0,\ldots,0).
\end{aligned}
$$

Call this sum S. The tensor products in this sum are indexed by three k-tuples of the form $\begin{pmatrix}i_1 & \ldots & i_k\\ j_1 & \ldots & j_k\\ l_1 & \ldots & l_k\end{pmatrix}$. For a given $(l'_1,\ldots,l'_{k+1})$, we will show that there is exactly one $\begin{pmatrix}i_1 & \ldots & i_k\\ j_1 & \ldots & j_k\\ l_1 & \ldots & l_k\end{pmatrix}$ such that, for $1\le n\le k$,

$$x_n-i_n+j_n+l_n-(n-1)=x_n+l'_n-(n-1)$$

and

$$(-1)^{i+j}\operatorname{sgn}\begin{pmatrix}0 & 1 & \ldots & k-1\\ |l_1| & |l_2| & \ldots & |l_k|\end{pmatrix}=\operatorname{sgn}\begin{pmatrix}0 & 1 & \ldots & k-1 & k\\ |l'_1| & |l'_2| & \ldots & |l'_k| & |l'_{k+1}|\end{pmatrix}.$$

We will now give an explicit description of how to calculate this $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ from $(l'_1, \dots, l'_{k+1})$, and with some thought, it is easy enough to see that this is the only way to choose the proper index.

For a particular $(l'_1, \dots, l'_{k+1})$, choose $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ in the following way. If $l'_{k+1} = k$, then $i + j = 0$ and $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ is equal to $\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ l'_1 & \dots & l'_k \end{pmatrix}$. If $l'_{k+1} = -k$, then $i + j = 2k$ and $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ is equal to $\begin{pmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \\ l'_1 & \dots & l'_k \end{pmatrix}$. If $|l'_s| = k$ for $s \neq k+1$, take $i_s = 1$, $j_s = 0$, and $l_s = -(k-1)$ if $l'_s = -k$ and take $i_s = 0$, $j_s = 1$, and $l_s = k-1$ if $l'_s = k$. Next consider $|l'_r| = k-1$ and if $r \neq k+1$ take $i_r = 1$, $j_r = 0$, and $l_r = -(k-2)$ if $l'_r = -(k-1)$ and take $i_r = 0$, $j_r = 1$, and $l_r = k-2$ if $l'_s = k-1$. Continue with this process until $|l'_{k+1}| = k - t$ for some $0 < t \leq k$. For the other entries in $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$, take $l_a = l'_a$, and take $i_a = 0$ and $j_a = 0$ if $l'_{k+1} = k - t > 0$ and take $i_a = 1$ and $j_a = 1$ if $l'_{k+1} = -(k-t) < 0$. Note that if $l'_{k+1} = 0$, all of the entries have already been determined by the earlier process. Now for a particular element in S' indexed by $(l'_1, \ldots, l'_{k+1})$, we have the same element appearing in S indexed by the corresponding $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ and with the same sign attached.

The symmetric group on k letters acts on the elements of S by permuting the columns of the index of an element, $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$. Each $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ that corresonds to an element in S' as described above is the result of a permutation applied to one of four types. These four types are indexed by the following.

1) $\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & \pm 1 & \ldots & \pm(k-1) \end{pmatrix}$

2) $\begin{pmatrix} 0 & \ldots & 0 & i_n & \ldots & i_k \\ 0 & \ldots & 0 & j_n & \ldots & j_k \\ 0 & \ldots & \pm(n-2) & \pm(n-1) & \ldots & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k$. Note that when $n = 1$, either $i_1 = 0$ and $j_1 = 1$ or $i_1 = 1$ and $j_1 = 0$.

3) $\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 0 & \pm 1 & \ldots & \pm(k-1) \end{pmatrix}$

4) $\begin{pmatrix} 1 & \dots & 1 & i_n & \dots & i_k \\ 1 & \dots & 1 & j_n & \dots & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k$. When $n = 1$, this coincides with the second type for $n = 1$.

Now we will prove by induction on k that $S = S'$ by showing that $S = S_1 + S_2$, where S_1 is a subsum containing only those elements corresponding to elements in the sum S', in other words S_1 is equal to the sum of all of the elements indexed by permutations of the four types of indices listed above, and $S_2 = S - S_1 = 0$. The case where $k = 1$ was shown earlier. In this case every term in S corresponded to a term in S' and there was no cancellation, so that $S_2 = 0$ trivially. The case where $k = 2$ was also explicitly calculated. Assume $S = S_1 + S_2$ such that $S_1 = S'$ and $S_2 = S - S_1 = 0$ for $k - 1$ and take $k > 2$. We want to show $S = S_1 + S_2$ such that $S_1 = S'$ and $S_2 = S - S_1 = 0$ for k.

Consider all $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ with a fixed $l_r = \pm(k-1)$ and fixed i_r and j_r.

Consider the following subsum contained in S,

$$S(r, \frac{l_r}{|l_r|}, i_r, j_r)$$

$$= \Big(\sum_{\substack{i_1,\ldots,\hat{i_r},\ldots,i_k \in \{0,1\} \\ j_1,\ldots,\hat{j_r},\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,\hat{l_r},\ldots,l_k \in \{0,\pm 1,\ldots,\pm(k-2)\} \\ \{|l_1|,\ldots,|\hat{l_r}|,\ldots,|l_k|\} = \{0,1,\ldots,k-2\}}} (-1)^{i+j} \mathrm{sgn} \begin{pmatrix} 0 & 1 & \ldots & k-1 \\ & & & \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix}$$

$$\bigotimes_{\substack{n=1 \\ n \neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0) \otimes V(z - (i+j), 0, \ldots, 0) \Big)$$

$$\otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0).$$

This sum is equal to

$$(-1)^{(k-r)}(-1)^{(i_r+j_r)} \Big(\sum_{\substack{i_1,\ldots,\hat{i_r},\ldots,i_k \in \{0,1\} \\ j_1,\ldots,\hat{j_r},\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,\hat{l_r},\ldots,l_k \in \{0,\pm 1,\ldots,\pm(k-2)\} \\ \{|l_1|,\ldots,|\hat{l_r}|,\ldots,|l_k|\} = \{0,1,\ldots,k-2\}}} (-1)^{i+j-i_r-j_r}$$

$$\mathrm{sgn} \begin{pmatrix} 0 & 1 & \ldots & & \ldots & k-2 \\ & & & & & \\ |l_1| & |l_2| & \ldots & |\hat{l_r}| & \ldots & |l_k| \end{pmatrix} \bigotimes_{\substack{n=1 \\ n \neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0)$$

$$\otimes V((z - i_r - j_r) - (i + j - i_r - j_r), 0, \ldots, 0) \Big)$$

$$\otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0)$$

$$= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0).$$

The sum R is equal to

$$\sum_{\substack{i_1,\dots,\hat{i_r},\dots,i_k\in\{0,1\}\\ j_1,\dots,\hat{j_r},\dots,j_k\in\{0,1\}}} \sum_{\substack{l_1,\dots,\hat{l_r},\dots,l_k\in\{0,\pm1,\dots,\pm(k-2)\}\\ \{|l_1|,\dots,|\hat{l_r}|,\dots,|l_k|\}=\{0,1,\dots,k-2\}}} (-1)^{i+j-i_r-j_r}$$
$$\operatorname{sgn}\begin{pmatrix} 0 & 1 & \dots & & \dots & k-2 \\ |l_1| & |l_2| & \dots & |\hat{l_r}| & \dots & |l_k| \end{pmatrix} \bigotimes_{\substack{n=1\\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \dots, 0)$$
$$\otimes V((z - i_r - j_r) - (i + j - i_r - j_r), 0, \dots, 0).$$

Apply the inductive hypothesis to R. By the inductive hypothesis for $k-1$, R is equal to $R_1 + R_2$ such that R_1 contains a sum of elements indexed by permutations (from the symmetric group on $\{1, \dots, r-1, r+1, \dots, k\}$) of the four special types, with the r-th column removed, and $R_2 = R - R_1 = 0$. Therefore, $R = R_1$.

Now

$$S(r, \frac{l_r}{|l_r|}, i_r, j_r)$$
$$= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \dots, 0)$$
$$= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R_1) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \dots, 0)$$
$$= \Big(\sum_{\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}} (-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \dots & k-1 \\ |l_1| & |l_2| & \dots & |l_k| \end{pmatrix}$$
$$\bigotimes_{\substack{n=1\\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \dots, 0) \otimes V(z - (i+j), 0, \dots, 0)\Big)$$
$$\otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \dots, 0)$$

for all $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ with the fixed r-th column and the rest of the matrix equal to a permutation (from the symmetric group on $\{1, \dots, r-1, r+1, \dots, k\}$) of one of the four types.

$$S = \sum_{\substack{r \in \{1,\dots,k\} \\ \epsilon \in \{-1,+1\} \\ i_r, j_r \in \{0,1\}}} S(r, \epsilon, i_r, j_r)$$

for the subsums $S(r, \epsilon, i_r, j_r)$, and all of these have been reduced by the inductive hypothesis.

We will show that for the remaining elements in the subsum $S_2 = S - S_1$, which have not been cancelled out by the application of the inductive hypothesis, there is a well-defined pairing of elements into disjoint pairs such that the sum of the elements in a pair is equal to zero. Notice that two indices $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ and $\begin{pmatrix} i'_1 & \dots & i'_k \\ j'_1 & \dots & j'_k \\ l'_1 & \dots & l'_k \end{pmatrix}$ will correspond to elements that will sum to zero if $-(i_1, \dots, i_k) + (j_1, \dots, j_k) + (l_1, \dots, l_k) = -(i'_1, \dots, i'_k) + (j'_1, \dots, j'_k) + (l'_1, \dots, l'_k)$, $i + j = i' + j'$, and the signs associated to $(l_1, \dots, l_k)$ and $(l'_1, \dots, l'_k)$ are different.

Define the set $\mathcal{M}$ to be the elements in the reduced $S_2 = S - S_1$. This means all of the elements indexed by a matrix $M = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ such that if the $l_r = \pm k - 1$, then M is not a permutation of one of the four special types of indices but with the r-th column removed it is a permutation of one of the four special types of indices (for $k-1$). This means any M is a permutation of one of the following.

1) $\begin{pmatrix} 0 & 0 & \dots & 0 & i_k \\ 0 & 0 & \dots & 0 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$. The last two columns are equal to one of the following

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \pm(k-2) & (k-1) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \pm(k-2) & -(k-1) \end{pmatrix}, \text{or} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ \pm(k-2) & \pm(k-1) \end{pmatrix}.$$

2) $\begin{pmatrix} 0 & \dots & 0 & i_n & \dots & i_{k-1} & i_k \\ 0 & \dots & 0 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$

for $n \le r \le k-1$. The last two columns are equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -(k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -(k-2) & (k-1) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -(k-2) & -(k-1) \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -(k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ (k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ (k-2) & (k-1) \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ (k-2) & -(k-1) \end{pmatrix}, \text{or} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ (k-2) & \pm(k-1) \end{pmatrix}.$$

3) $\begin{pmatrix} 1 & 1 & \dots & 1 & i_k \\ 1 & 1 & \dots & 1 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$. The last two columns are equal to one of the following,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ \pm(k-2) & (k-1) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \pm(k-2) & -(k-1) \end{pmatrix}, \text{or} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \pm(k-2) & \pm(k-1) \end{pmatrix}.$$

4) $\begin{pmatrix} 1 & \dots & 1 & i_n & \dots & i_{k-1} & i_k \\ 1 & \dots & 1 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k-1$. The last two columns are equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -(k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -(k-2) & (k-1) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -(k-2) & -(k-1) \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -(k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ (k-2) & \pm(k-1) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ (k-2) & (k-1) \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ (k-2) & -(k-1) \end{pmatrix}, \text{or} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ (k-2) & \pm(k-1) \end{pmatrix}.$$

Now define a function $\Xi : \mathcal{M} \to \mathcal{M}$. We will define it for elements with these four types of indices in terms of their indices. Then the definition for any other element can be found by endowing Ξ with the property that $\Xi(\sigma M) = \sigma\Xi(M)$ for any $\sigma \in S_k$ and any index M of an element in $\mathcal{M}$. Ξ is now defined for all elements in $\mathcal{M}$ because any index of an element can be found as a permutation of one of these four types of indices.

1) Given indices of the form $\begin{pmatrix} 0 & 0 & \dots & 0 & i_k \\ 0 & 0 & \dots & 0 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$, define Ξ in the following way.

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

2) Given indices of the form

$$\begin{pmatrix} 0 & \dots & 0 & i_n & \dots & i_{k-1} & i_k \\ 0 & \dots & 0 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$$

for some $1 \leq n \leq k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k-1$, define the Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n = k-1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n > k-1$, either $i_{k-2} = 1$, $j_{k-2} = 0$, and $l_{k-2} = -(k-3)$ or $i_{k-2} = 0$, $j_{k-2} = 1$, and $l_{k-2} = k-3$. Define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & -(k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & -(k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & (k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & (k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & (k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & (k-1) & \dots & -(k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & -(k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & -(k-1) & \dots & -(k-3) \end{pmatrix}$$

3) Given indices of the form $\begin{pmatrix} 1 & 1 & \dots & 1 & i_k \\ 1 & 1 & \dots & 1 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$, define Ξ in the following way.

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

4) Given indices of the form

$$\begin{pmatrix} 1 & \dots & 1 & i_n & \dots & i_{k-1} & i_k \\ 1 & \dots & 1 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$$

for some $1 \leq n \leq k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k-1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n = k - 1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n > k-1$, either $i_{k-2} = 1$, $j_{k-2} = 0$, and $l_{k-2} = -(k-3)$ or $i_{k-2} = 0$, $j_{k-2} = 1$, and $l_{k-2} = k-3$. Define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & -(k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & -(k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & (k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & (k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & (k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & (k-1) & \dots & -(k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & -(k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & -(k-1) & \dots & -(k-3) \end{pmatrix}$$

Ξ is well-defined because it sends any element in $\mathcal{M}$ to another such element. This is obviously true for the elements with the particular indices Ξ was explicitly defined for, and since $\mathcal{M}$ is invariant under S_k, this is true for all elements in $\mathcal{M}$. It is also easy enough to verify that $\Xi^2 = Id$. Let $[M]$ be the element indexed by the matrix M, and let $\text{sgn}(M)$ be the sign associated to that element. Then $S_2 = \sum_{M \in \mathcal{M}} [M]$. Also, $|\{M \in \mathcal{M} \,|\, \text{sgn}(M) = 1\}| = |\{M \in \mathcal{M} \,|\, \text{sgn}(M) = -1\}|$ and $\text{sgn}(\Xi(M)) = -\text{sgn}(M)$. Therefore Ξ is a bijection between $\{M \in \mathcal{M} \,|\, \text{sgn}(M) =$

1} and $\{M \in \mathcal{M} \,|\, \mathrm{sgn}(M) = -1\}$.

$$S_2 = \sum_{\substack{M\in\mathcal{M}\\ \mathrm{sgn}(M)=1}} [M] + \sum_{\substack{M\in\mathcal{M}\\ \mathrm{sgn}(M)=1}} [\Xi(M)] = \sum_{\substack{M\in\mathcal{M}\\ \mathrm{sgn}(M)=1}} ([M] + [\Xi(M)])$$

We claim that $[M] + [\Xi(M)] = 0$ for every $M \in \mathcal{M}$. Again, we only need to consider M as one of the four special types because for any other index σM for some $\sigma \in S_k$ will have $[\sigma M] + [\Xi(\sigma M)] = [\sigma M] + [\sigma \Xi(M)] = \sigma([M] + [\Xi(M)]) = \sigma(0) = 0$.

To see $[M] + [\Xi(M)] = 0$ for some M that is one of the four special types, it is enough to show M and $\Xi(M)$ satisfy the three necessary conditions. For $M = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ and $\Xi(M) = \begin{pmatrix} i'_1 & \dots & i'_k \\ j'_1 & \dots & j'_k \\ l'_1 & \dots & l'_k \end{pmatrix}$, it is an easy calculation to check

$$\begin{aligned} &-(i_1, \dots, i_k) + (j_1, \dots, j_k) + (l_1, \dots, l_k) \\ &= -(i'_1, \dots, i'_k) + (j'_1, \dots, j'_k) + (l'_1, \dots, l'_k). \end{aligned}$$

Also, $\Xi(M)$ does not change the number of entries equal to 1 in the i_r and j_s slots for M. Therefore $i + j = i' + j'$. Also, $\Xi(M)$ involves a transposition of $(l_1, \dots, l_k)$, so the signs associated to the elements are different.

Therefore, $S_2 = \sum_{\substack{M\in\mathcal{M}\\ \mathrm{sgn}(M)=1}} 0 = 0$. This completes the proof.

□

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DEDICATION

to

Mathematics,

My family,

and Anyone who's bothering to read this

For

The logical, the illogical, and everything in-between.