Representations as formal sums of symmetric tensor spaces for symplectic Lie algebras

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Abstract

Any irreducible representation of $\mathfrak{sp}(2m,\mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation. First I will present a basic result from linear algebra, which lays the foundation for an initial case of this statement. Then I will prove the general statement, which involves the application of Littelmann's work with Young diagrams in the context of decomposing a tensor product of two irreducible representations of $\mathfrak{sp}(2m,\mathbb{C})$. Finally I will set up an algorithm for finding such sums and provide some of the consequences in the cases of $\mathfrak{sp}(4,\mathbb{C})$ and $\mathfrak{sp}(6,\mathbb{C})$ and two general formulas.

Keywords: multilinear algebra; symmetric tensors; $\mathfrak{sp}(2m, \mathbb{C})$; $\mathrm{Sp}(2m)$; C_m; symplectic Lie algebra; Littlewood-Richardson rule; Littelmann

1 Introduction

Finite-dimensional representations of classical Lie algebras have been studied since the 19th century, and have found many applications in Mathematics and Physics. It is therefore surprising that basic questions remain unanswered, even in low-dimensional cases. For example, an explicit formula for the weight multiplicities in finite-dimensional representations of $\mathfrak{sp}(4, \mathbb{C})$ has only been published in 2004 in [3].

Any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation. This is the main result of this note along with an algorithm for determining such formal sums and two formulas.

In [1] and [8], the authors, Akin and Zelevinskii respectively, independently prove an identity expressing any irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ as a formal sum of tensor products of symmetric powers of the standard representation using resolutions. We will present a different approach for Lie algebras of type C_m.

We first present a useful identity between finite-dimensional representations of the rank m symplectic Lie algebra. In Sections 2 and 3, using an elementary approach, we develop this first identity. It is based on a general result involving multilinear algebra for symmetric tensors; see Proposition 2.1 and Corollary 2.2. While these are certainly well known to experts, we have included proofs for completeness. Our first result, Theorem 3.1 (and subsequently Corollary 3.2), follows from this together with the explicit determination of certain highest weight vectors occurring in a tensor product of symmetric powers of the standard representation of $\mathfrak{sp}(2m, \mathbb{C})$. Corollary 3.2 then shows how an irreducible representation can be expressed as a linear combination of tensor products of symmetric powers of the standard representation.

In Section 4, we use Littelmann's generalization of the Littlewood-Richardson rule in [7] and apply it to $\mathrm{Sp}(2m)$ to prove the main result of this note, Theorem 4.2. This theorem states that any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation, and the method of proof creates an algorithm for finding such a sum. In Section 5, we present a refinement of the algorithm from the proof along with a formula, which simplifies the process for finding the formal sum.

In Section 6, we show examples for the symplectic Lie algebras of rank 2 and 3. At the end of this note in Section 7, we present a final formula that explicitly determines the formal sum for a general case.

The original motivation for these identifications was to calculate L- and ε-factors for representations of the real Weil group, which requires precise multiplicity information. However, we hope that the results of this paper are of independent interest beyond this immediate application.

2 A result on symmetric tensors

For a positive integer n, let S_n be the symmetric group on n letters. For this section, let V be a finite-dimensional vector space over a field with characteristic zero, F. S_n acts linearly on $V^{\otimes n}$ by $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$. Let $\mathrm{sym} : V^{\otimes n} \to V^{\otimes n}$ be the usual symmetrization map, i.e., $\mathrm{sym}(v) = \sum_{\sigma \in S_n} \sigma(v)$. The kernel of this map is spanned by all elements of the form $v - \sigma(v)$ for $v \in V^{\otimes n}$ and $\sigma \in S_n$. We denote by $\mathrm{Sym}^n(V)$ the image of sym or equivalently the quotient of $V^{\otimes n}$ by the kernel of sym.

Let V have the basis $\{v_1, v_2, \ldots, v_k\}$, and let V^* be the dual space with corresponding dual basis $\{f_1, f_2, \ldots, f_k\}$. Let W be another finite-dimensional vector space over F with the basis $\{w_1, w_2, \ldots, w_l\}$. Using the given bases of V and W, we identify the standard basis elements of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$ with pairs of k- and l-tuples $(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)$ such that, for a particular basis element, c_i equals the number of times v_i appears in that basis element and d_j equals the number of times w_j appears in that basis element. When $W = V^*$, $l = k$ and the standard basis elements of $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m V^*$ are identified with pairs of k-tuples such that c_i is as before and d_j equals the number of times f_j appears in the basis element. The standard basis for $\mathrm{Sym}^n V \otimes \mathrm{Sym}^m W$ is then given by

$$\{(c_1, \ldots, c_k) \times (d_1, \ldots, d_l) \mid c_i \in \mathbb{Z}_{\geq 0}, d_j \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{k} c_i = n, \sum_{j=1}^{l} d_j = m\}.$$

For $x, y \geq 1$, consider the linear map $\rho : \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V^* \to \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V^*$ with the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{x-1}) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{y-1}) \\ &\quad \longmapsto \sum_{i=1}^{k} \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{x-1} \otimes v_i) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{y-1} \otimes f_i). \end{aligned}$$

This is the map defined as multiplication by the element $\sum_{i=1}^{k} v_i \otimes f_i$, which generates the trivial representation in $V \otimes V^*$. The following shows ρ is an injective intertwining map.

Proposition 2.1. *Let V and W be finite-dimensional representations of a Lie algebra L, such that there is a trivial representation contained in $V \otimes W$. For integers $x, y \geq 1$, let $\phi : \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} W \to \mathrm{Sym}^x V \otimes \mathrm{Sym}^y W$ be the linear map defined as multiplication by some fixed generator of the trivial representation. Then ρ is an injective intertwining map.*

Proof. Given V and W are representations with the previously-defined bases, let $\sum_{i,j} a_{ij} v_i \otimes w_j$, for some coefficients a_{ij}, generate a trivial representation in $V \otimes W$, and assume without loss of generality $a_{11} \neq 0$. Then ρ becomes the linear map defined by the property

$$\begin{aligned} &\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{x-1}) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{y-1}) \\ &\quad \longmapsto \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_{x-1} \otimes v_i) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_{y-1} \otimes w_j). \end{aligned}$$

To prove injectivity directly, first consider an element of the kernel of ρ written as a linear combination of basis vectors $(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)$ of $\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} W$.

$$\mathbf{v} = \sum_{(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)} b_{(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)} (c_1, \ldots, c_k) \times (d_1, \ldots, d_l)$$

for some constants $b_{(c_1, \ldots, c_k) \times (d_1, \ldots, d_l)}$. Then, $\rho(\mathbf{v}) = 0$ in $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y W$, and $\rho(\mathbf{v})$ written as a linear combination of basis vectors $(e_1, \ldots, e_k) \times (f_1, \ldots, f_l)$ of $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y W$ will have its coefficient of $(e_1, \ldots, e_k) \times (f_1, \ldots, f_l)$ equal to

$$\sum_{\substack{i,j \\ e_i, f_j \neq 0}} a_{ij} b_{(e_1, \ldots, e_i - 1, \ldots, e_k) \times (f_1, \ldots, f_j - 1, \ldots, f_l)} = 0.$$

We now note that each element of the standard basis of $\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} W$ is of the form $(x - r_1, c_2, \ldots, c_k) \times (y - r_2, d_2, \ldots, d_l)$ for some r_1 such that $1 \leq r_1 \leq x$ and some r_2 such that $1 \leq r_2 \leq y$, with $c_i, d_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=2}^{k} c_i = r_1 - 1$, and $\sum_{i=2}^{l} d_i = r_2 - 1$. It is then straightforward to prove by induction on $s = r_1 + r_2$ that each of the coefficients $b_{(x - r_1, c_2, \ldots, c_k) \times (y - r_2, d_2, \ldots, d_l)} = 0$ thus proving $\ker(\rho) = \{0\}$.

To prove injectivity an alternate way, let $\mathrm{Sym}(V)$ be the algebra $\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V$. Then $\mathrm{Sym}(V) \otimes \mathrm{Sym}(W) = \mathrm{Sym}(V + W)$ is isomporphic to the set of polynomials on $(V^* + W^*)$

over F, which has no zero divisors. This implies ρ is injective because in this setting, ρ is equivalent to multiplying certain homogeneous degree $x-1+y-1$ polynomials by a fixed homogeneous degree 2 polynomial.

The intertwining property of ρ is easy to verify using the fact that $\sum_{i,j} a_{ij} v_i \otimes w_j$ generates a trivial representation in $V \otimes W$. This concludes the proof. □

The dual map to ρ (with x and y interchanged) is the linear map $\rho^* : \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V^* \to \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V^*$ with the property

$$\begin{aligned}&\mathrm{sym}(\alpha_1 \otimes \ldots \otimes \alpha_x) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \beta_y)\\ &\quad\longmapsto \sum_{i=1}^{x} \sum_{j=1}^{y} \beta_j(\alpha_i) \mathrm{sym}(\alpha_1 \otimes \ldots \otimes \hat{\alpha_i} \otimes \ldots \otimes \alpha_x) \otimes \mathrm{sym}(\beta_1 \otimes \ldots \otimes \hat{\beta_j} \otimes \ldots \otimes \beta_y).\end{aligned}$$

ρ^* is a surjective intertwining map.

We obtain the following result from Proposition 2.1.

Corollary 2.2. *Let V be a finite-dimensional representation of a Lie algebra. Then there exists an invariant subspace*

$$\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V^* \subset \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V^* \text{ for all integers } x, y \geq 1.$$

3 The case of $\mathfrak{sp}(2m, \mathbb{C})$

We will apply the above result of Corollary 2.2 to representations of the Lie algebra

$$\mathfrak{sp}(2m, \mathbb{C}) = \{A \in \mathfrak{gl}(2m, \mathbb{C}) \mid A^t J + JA = 0\}.$$

Here $J = \left[\begin{smallmatrix} 0 & J_m \\ -J_m & 0 \end{smallmatrix}\right]$ and J_m is defined to be the $m \times m$ anti-diagonal matrix with ones along the anti-diagonal. Evidently, $\mathfrak{sp}(2m, \mathbb{C})$ is $(2m^2 + m)$-dimensional and has the following basis,

$$\{H_k\} = \{e_{kk} - e_{2m+1-k,2m+1-k} | k = 1, \ldots, m\},$$

$$\begin{aligned}\{X_\alpha\} = \{&e_{ij} - e_{2m+1-j,2m+1-i} | (i,j) = (1,2), \ldots, (1,m), (2,3), \ldots, (2,m), \ldots, (m-1,m)\}\\ &\cup \{e_{i,2m+1-j} + e_{j,2m+1-i} | (i,j) = (1,2), \ldots, (1,m), (2,3), \ldots, (2,m), \ldots, (m-1,m)\}\\ &\cup \{e_{i,2m+1-i} | i = 1, \ldots, m\},\end{aligned}$$

and

$$\{Y_\alpha\} = \{Y_\alpha = X_\alpha^t\}.$$

In this basis, the Cartan subalgebra is $\mathfrak{h} = \langle H_1, \ldots, H_m \rangle$, and for each root α,

$$\mathfrak{s}^\alpha = \mathrm{span}\{X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Any weight $(x_1, x_2, \ldots, x_m)$ can be thought of as the eigenvalues associated to H_1 through H_m, respectively, for the corresponding weight vector. The weights in the dominant Weyl chamber are $\{(x_1, \ldots, x_m) \in \mathbb{Z}^m : x_1 \geq x_2 \geq \ldots \geq x_m \geq 0\}$. Let $V(x_1, \ldots, x_m)$ be the irreducible representation with highest weight $(x_1, \ldots, x_m)$.

The Weyl dimension formula, tailored to our situation, appears in [5], Section 7.6.3. It states that

$$\dim V(x_1, \ldots, x_m) = \frac{\prod_{\alpha > 0} wt(X_\alpha) \cdot ((x_1, \ldots, x_m) + wt(\delta))}{\prod_{\alpha > 0} wt(X_\alpha) \cdot wt(\delta)}$$

where $wt(X_\alpha)$ is the weight of X_α in the adjoint representation, $\cdot$ is the normal dot product, δ is half of the sum of the positive roots, and $wt(\delta)$ is the weight of δ in the adjoint representation.

The standard representation of $\mathfrak{sp}(2m,\mathbb{C})$ is $V(1,0,\dots,0)$. It has the standard basis $\{e_1,\dots,e_{2m}\}$ and is isomporphic to its dual representation with corresponding basis $\{f_1,\dots,f_{2m}\}$. These representations are isomorphic via $f_i \mapsto e_{2m+1-i}$ for $m+1 \le i \le 2m$ and $f_j \mapsto -e_{2m+1-j}$ for $1 \le j \le m$. The weights of $V(1,0,\dots,0)$ are

$$\{(1,0,\dots,0),\dots,(0,\dots,1),(0,\dots,0,-1),\dots,(-1,0,\dots,0)\},$$

and e_1 is a highest weight vector.

It can be easily shown that $V(n,0,\dots,0) = \mathrm{Sym}^n V(1,0,\dots,0)$. First, there is a highest weight vector, $\mathrm{sym}(e_1 \otimes \dots \otimes e_1)$, in $\mathrm{Sym}^n V(1,0,\dots,0)$ with weight $(n,0,\dots,0)$, and therefore $V(n,0,\dots,0) \subset \mathrm{Sym}^n V(1,0,\dots,0)$. Then using the Weyl dimension formula, $V(n,0,\dots,0)$ has the same dimension as $\mathrm{Sym}^n V(1,0,\dots,0)$ and thus $V(n,0,\dots,0) = \mathrm{Sym}^n V(1,0,\dots,0)$.

Theorem 3.1. *For $\mathfrak{sp}(2m,\mathbb{C})$ and its standard representation $V = V(1,0,\dots,0)$,*

$$\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V = (\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V) \oplus \bigoplus_{p=0}^{y} V(x+y-p,p,0,\dots,0)$$

for integers $x \ge y \ge 1$.

Proof. Given $x \ge y$ and using the previously described basis, we define for all integers p such that $0 \le p \le y$ the following vector in $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V^*$,

$$\begin{aligned} v_p &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i (x-p+i, p-i, 0,\dots,0) \times (0,\dots,0,i,y-i) \\ &= \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mathrm{sym}(\underbrace{e_1 \otimes \dots \otimes e_1}_{x-p+i} \otimes \underbrace{e_2 \otimes \dots \otimes e_2}_{p-i}) \\ &\qquad \otimes \mathrm{sym}(\underbrace{f_{2m-1} \otimes \dots \otimes f_{2m-1}}_{i} \otimes \underbrace{f_{2m} \otimes \dots \otimes f_{2m}}_{y-i}). \end{aligned}$$

This vector is in the kernel of the map ρ^* defined in Section 2 because $(x-p+i, p-i, 0,\dots,0) \times (0,\dots,0,i,y-i) \mapsto 0 + \dots + 0 = 0$. Also, this vector is a highest weight vector with weight

$$\begin{aligned} &(x-p+i)(1,0,\dots,0) + (p-i)(0,1,0,\dots,0) \\ &\qquad + i(0,1,0,\dots,0) + (y-i)(1,0,\dots,0) = (x+y-p,p,0,\dots,0). \end{aligned}$$

To see v_p is a highest weight vector, it is enough to show that it is in the kernel of X_α for any α.

First, the only relevant calculations are $X_\alpha.e_1$, $X_\alpha.e_2$, $X_\alpha.f_{2m}$, and $X_\alpha.f_{2m-1}$. These will all be equal to zero except when $X_\alpha = e_{12} - e_{2m-1,2m}$. Therefore, we only need to show v_p is in the kernel of $X_\alpha = e_{12} - e_{2m-1,2m}$. Call this root X_{12}. $X_{12}.e_1 = X_{12}.f_{2m} = 0$, $X_{12}.e_2 = e_1$, and $X_{12}.f_{2m-1} = f_{2m}$. By definition,

$$\begin{aligned} &X_{12}.(x-p+i, p-i, 0,\dots,0) \times (0,\dots,0,i,y-i) \\ &\quad = X_{12}.\mathrm{sym}(\underbrace{e_1 \otimes \dots \otimes e_1}_{x-p+i} \otimes \underbrace{e_2 \otimes \dots \otimes e_2}_{p-i}) \otimes \mathrm{sym}(\underbrace{f_{2m-1} \otimes \dots \otimes f_{2m-1}}_{i} \otimes \underbrace{f_{2m} \otimes \dots \otimes f_{2m}}_{y-i}). \end{aligned}$$

This becomes

$$\begin{aligned}
&(x-p+i)\mathrm{sym}(X_{12}.e_1 \otimes e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2)\\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \ldots \otimes f_{2m-1} \otimes f_{2m} \otimes \ldots \otimes f_{2m})\\
&+(p-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes X_{12}.e_2 \otimes e_2 \otimes \ldots \otimes e_2)\\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \ldots \otimes f_{2m-1} \otimes f_{2m} \otimes \ldots \otimes f_{2m})\\
&+(i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2)\\
&\qquad \otimes \mathrm{sym}(X_{12}.f_{2m-1} \otimes f_{2m-1} \otimes \ldots \otimes f_{2m-1} \otimes f_{2m} \otimes \ldots \otimes f_{2m})\\
&+(y-i)\mathrm{sym}(e_1 \otimes \ldots \otimes e_1 \otimes e_2 \otimes \ldots \otimes e_2)\\
&\qquad \otimes \mathrm{sym}(f_{2m-1} \otimes \ldots \otimes f_{2m-1} \otimes X_{12}.f_{2m} \otimes f_{2m} \otimes \ldots \otimes f_{2m}).
\end{aligned}$$

This is equal to $(x-p+i)(0)+(p-i)(x-p+i+1,p-i-1,0,\ldots,0)\times(0,\ldots,0,i,y-i)+(i)(x-p+i,p-i,0,\ldots,0)\times(0,\ldots,0,i-1,y-i+1)+(y-i)(0)$ (with the understanding that when $i=p$ there is no second term and when $i=0$ there is no third term here). From here $X_{12}.v_p=0$ is a straightforward calculation.

For each of these highest weight vectors, v_p, with weight $(x+y-p,p,0\ldots,0)$ and in the kernel of ρ^*, there is an irreducible representation $V(x+y-p,p,0,\ldots,0)$ contained in the kernel. Since all of the weights $\{(x+y-p,p,0\ldots,0) : 0 \le p \le y\}$, are distinct, $\bigoplus_{p=0}^{y} V(x+y-p,p,0,\ldots,0) \subset \ker(\rho^*)$.

It follows from semisimplicity and the surjectivity of ρ^* that

$$\begin{aligned}
&(\mathrm{Sym}^{x-1}V \otimes \mathrm{Sym}^{y-1}V^*) \oplus \bigoplus_{p=0}^{y} V(x+y-p,p,0,\ldots,0)\\
&\quad \subset (\mathrm{Sym}^{x-1}V \otimes \mathrm{Sym}^{y-1}V^*) \oplus \ker(\rho^*)\\
&\quad = \mathrm{Sym}^{x}V \otimes \mathrm{Sym}^{y}V^*
\end{aligned}$$

for $x \ge y \ge 1$. The Weyl dimension formula shows that this inclusion is actually an equality. Note that V^* can be replaced by V since this representation is self-dual. □

Note that all of the highest weight vectors in $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V$, $V = V(1,0,\ldots,0)$, can be determined using the proof of Theorem 3.1, the map ρ from Section 2, and the isomorphism between the standard representation and its dual.

Corollary 3.2. *For integers $x \ge y = 1$,*

$$V(x,0,\ldots,0) \otimes V(1,0,\ldots,0) = V(x+1,0,\ldots,0) \oplus V(x,1,0,\ldots,0) \oplus V(x-1,0,\ldots,0)$$

For $x \ge y \ge 2$,

$$\begin{aligned}
&(V(x,0,\ldots,0) \otimes V(y,0,\ldots,0)) \oplus (V(x,0,\ldots,0) \otimes V(y-2,0,\ldots,0))\\
&\quad = (V(x+1,0,\ldots,0) \otimes V(y-1,0,\ldots,0)) \oplus V(x,y,0,\ldots,0)\\
&\qquad \oplus (V(x-1,0,\ldots,0) \otimes V(y-1,0,\ldots,0))
\end{aligned}$$

Proof. Recall $\mathrm{Sym}^n V(1,0,\ldots,0) = V(n,0,\ldots,0)$. The first assertion is the special case of Theorem 3.1 where $y=1$. Using Theorem 3.1, when $x \ge y \ge 2$,

$$\begin{aligned}
&V(x,0,\ldots,0) \otimes V(y,0,\ldots,0)\\
&\quad = (V(x-1,0,\ldots,0) \otimes V(y-1,0,\ldots,0)) \oplus \bigoplus_{p=0}^{y} V(x+y-p,p,0,\ldots,0)
\end{aligned}$$

and

$$\begin{aligned}
&V(x+1,0,\ldots,0) \otimes V(y-1,0,\ldots,0)\\
&\quad = (V(x,0,\ldots,0) \otimes V(y-2,0,\ldots,0)) \oplus \bigoplus_{p=0}^{y-1} V(x+y-p,p,0,\ldots,0).
\end{aligned}$$

Combining these equations yields the assertion. □

In the Grothendieck group of all representations of $\mathfrak{sp}(2m,\mathbb{C})$, for $V = V(1,0,\ldots,0)$, we get

$$\begin{aligned}
V(x,0,\ldots,0) &= \mathrm{Sym}^x V\\
V(x,1,0,\ldots,0) &= \mathrm{Sym}^x V \otimes V - \mathrm{Sym}^{x+1} V - \mathrm{Sym}^{x-1} V\\
V(x,y,0,\ldots,0) &= \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V \otimes \mathrm{Sym}^{y-2} V\\
&\quad - \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V - \mathrm{Sym}^{x+1} V \otimes \mathrm{Sym}^{y-1} V.
\end{aligned}$$

4 The Littlewood-Richardson rule for $\mathrm{Sp}(2m)$

This idea of using the standard representation as a building block for determining every irreducible representation can then be expanded to more complicated highest weights, but more machinery is needed. In [7], Littelmann provides a generalization of the Littlewood-Richardson rule in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. The main result from [7] is the following theorem.

Theorem 4.1. *The decomposition of the tensor product $V_\lambda \otimes V_\mu$ into irreducible G-modules is given by*

$$V_\lambda \otimes V_\mu = \bigoplus_{\mathrm{T}} V_{\lambda+v(\mathrm{T})}$$

where T *runs over all G-standard Young tableaux of shape $p(\mu)$ that are λ-dominant.*

Let $G = \mathrm{Sp}(2m)$. We will now give a description, tailored to our situation, of the $\mathrm{Sp}(2m)$-standard Young tableaux of shape $p(\mu)$ that are λ-dominant. We will only need to consider the case where $\mu = (n,0,\ldots,0)$.

The $\mathrm{Sp}(2m)$-standard Young tableaux of shape $p(n,0,\ldots,0)$ are all of the Young diagrams consisting of a single nondecreasing column of length n containing the integers 1 to $2m$.

Define $v(\mathrm{T}) := (c_{\mathrm{T}}(1)-c_{\mathrm{T}}(2m))\epsilon_1+(c_{\mathrm{T}}(2)-c_{\mathrm{T}}(2m-1))\epsilon_2+\ldots+(c_{\mathrm{T}}(m)-c_{\mathrm{T}}(m+1))\epsilon_m$, where $c_{\mathrm{T}}(i)$ is equal to the number of times the number i appears in the tableau T.

Let $\mathrm{T}(l)$ be the tableau created from T by removing rows $l+1$ to n, counting from bottom to top. Then an $\mathrm{Sp}(2m)$-standard Young tableau T of shape $p(n,0,\ldots,0)$ is λ-dominant if all of the weights $\lambda + v(\mathrm{T}(l))$ are dominant weights for $1 \le l \le n$.

We can now present the main theorem of this note.

Theorem 4.2. *Any irreducible representation $V(x_1,\ldots,x_k,0,\ldots,0)$ of $\mathfrak{sp}(2m,\mathbb{C})$ can be written as an integral combination in the form*

$$\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_k^{(i)},0,\ldots,0) \tag{1}$$

for some n, c_i, and $y_j^{(i)}$.

Proof. We will use induction on k. The case where $k=1$ is trivial, and the case where $k=2$ is a consequence of Corollary 3.2.

Assume the statement of the theorem is true for k. Now, we will want to show $V(x_1,\ldots,x_k,x_{k+1},0,\ldots,0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$ for some n, c_i, and $y_j^{(i)}$. We will prove this assertion by induction on the size of x_{k+1}.

When $x_{k+1}=0$, $V(x_1,\ldots,x_k,x_{k+1},0,\ldots,0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$, with $y_{k+1}^{(i)} = 0$ for every i, using the inductive hypothesis for the induction on k.

Assume true for $x_{k+1} \leq z-1$. Now we want to show $V(x_1,\ldots,x_k,z,0,\ldots,0)$, for any x_i and z such that $x_1 \geq \ldots \geq x_k \geq z$, can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$ for some n, c_i, and $y_j^{(i)}$. Consider the decomposition of $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$ using Theorem 4.1.

The standard Young tableaux, T, of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant are the nondecreasing columns of length z with entries taken from the set of integers between 1 and $k+1$ and integers between $2m+1-k$ and $2m$ such that the following inequalities are satisfied for $1 \leq i \leq k-1$:

$$x_i - c_{\mathrm{T}}(2m+1-i) \geq x_{i+1} \tag{2}$$

$$x_i - c_{\mathrm{T}}(2m+1-i) \geq x_{i+1} + c_{\mathrm{T}}(i+1) - c_{\mathrm{T}}(2m-i) \tag{3}$$

$$x_k - c_{\mathrm{T}}(2m+1-k) \geq c_{\mathrm{T}}(k+1) \tag{4}$$

In the decomposition of $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$ using Theorem 4.1, each irreducible representation has a highest weight

$$(x_1 + c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m), \ldots, x_k + c_{\mathrm{T}}(k) - c_{\mathrm{T}}(2m+1-k), c_{\mathrm{T}}(k+1), 0, \ldots, 0)$$

for some standard Young tableaux, T, of shape $p(z,0,\ldots,0)$, which is $(x_1,\ldots,x_k,0,\ldots,0)$-dominant. All of these highest weights have $0 \leq c_{\mathrm{T}}(k+1) \leq z-1$ except in the case where T, a column of length z, contains only entries equal to $k+1$ with $c_{\mathrm{T}}(k+1) = z$. In this case, the highest weight is $(x_1,x_2,\ldots,x_k,z,0,\ldots,0)$. By induction, every other irreducible representation in the decomposition, except for $V(x_1,\ldots,x_k,z,0,\ldots,0)$, can be written as a some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$. $V(x_1,\ldots,x_k,0,\ldots,0)$ can be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_k^{(i)},0,\ldots,0)$ by the inductive hypothesis for the induction on k, so that $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$ is equivalent to $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_k^{(i)},0,\ldots,0) \otimes V(z,0\ldots,0)$.

By isolating $V(x_1,\ldots,x_k,z,0,\ldots,0)$ in the decomposition of $V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(z,0,\ldots,0)$, $V(x_1,\ldots,x_k,z,0,\ldots,0)$ can now be written as some integral combination in the form $\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0) \otimes \ldots \otimes V(y_{k+1}^{(i)},0,\ldots,0)$. This completes the induction on z and thus the induction on k.

□

This proof provides a recursive algorithm for finding the formal combination as in (1) for any irreducible representation. For example, the first step in determining (1) for $V(x_1,\ldots,x_k,1,0,\ldots,0)$ is the following identification:

$$\begin{aligned} V(x_1,\ldots,x_k,1,0,\ldots,0) = V(x_1,\ldots,x_k,0,\ldots,0) \otimes V(1,0,\ldots,0) \\ -\Big(\bigoplus_{\substack{x_{i-1}\neq x_i \\ i=1,\ldots,k}} V(x_1,\ldots,x_i+1,\ldots,x_k,0,\ldots,0)\Big) \\ -\Big(\bigoplus_{\substack{x_i\neq x_{i+1} \\ i=1,\ldots,k}} V(x_1,\ldots,x_i-1,\ldots,x_k,0,\ldots,0)\Big) \end{aligned}$$

5 A refinement of the recursive algorithm

For $V(x_1,\ldots,x_{k+1},0,\ldots,0)$, assume $z = x_{k+1} \geq 2k$ and all of the representations of the form $V(x_1,\ldots,x_k,0,\ldots,0)$ have known integral combinations in the form of (1). Define the following algorithm.

For $x_1 \neq x_2$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m)$$
$$:= V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0)$$
$$- V(x_1-1,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0).$$

If $x_1 = x_2$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m)$$
$$:= V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0).$$

For $x_i \neq x_{i+1}$ with $2 \leq i \leq k-1$ or $x_i \geq x_{i+1}$ with $i = k$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m+1-i)$$
$$:= F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m+2-i)$$
$$- F(V(x_1,\dots,x_i-1,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0),2m+2-i).$$

For $x_i = x_{i+1}$, $2 \leq i \leq k-1$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m+1-i)$$
$$:= F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m+2-i).$$

For $x_i \neq x_{i-1}$, $2 \leq i \leq k$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i)$$
$$:= F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i+1)$$
$$- F(V(x_1,\dots,x_i+1,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0),i+1).$$

For $x_i = x_{i-1}$, $2 \leq i \leq k$, define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i)$$
$$:= F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i+1).$$

Define the representation

$$F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),1)$$
$$:= F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2)$$
$$- F(V(x_1+1,x_2,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0),2).$$

Then,

$$V(x_1,\dots,x_k,z,0,\dots,0) = F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),1).$$

This algorithm produces an integral combination equal to $V(x_1,\dots,x_k,z,0,\dots,0)$ of representations of the form $V(x'_1,\dots,x'_k,0,\dots,0)\otimes V(z',0,\dots,0)$. Substituting in the integral combinations in the form of (1) for the representations $V(x'_1,\dots,x'_k,0,\dots,0)$ yields the integral combination in the form of (1) for $V(x_1,\dots,x_k,z,0,\dots,0)$. The following is an explanation of how the algorithm works.

Recall that $V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0) = \bigoplus_{\mathrm{T}} V((x_1,\dots,x_k,0,\dots,0)+v(\mathrm{T}))$ for all standard Young tableaux, T, of shape $p(z,0,\dots,0)$ that are $(x_1,\dots,x_k,0,\dots,0)$-dominant, which means all single nondecreasing columns, T, of length z containing integers from the set of integers between 1 and $k+1$ and integers between $2m+1-k$ and $2m$ and satisfying conditions (2), (3), and (4).

If $x_i \neq x_{i+1}$, define a map from the standard Young tableaux, $\tilde{\mathrm{T}}$, of shape $p(z-1,0,\dots,0)$ that are $(x_1,\dots,x_i-1,\dots,x_k,0,\dots,0)$-dominant and do not contain integers from the set $\{2m+2-i, \ldots, 2m\}$ to the standard Young tableaux, T, of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain integers from the set $\{2m+2-i, \ldots, 2m\}$ by sending $\tilde{\mathrm{T}}$ to the tableau formed by adding a $2m+1-i$ to the bottom of the column.

This map is a bijection between all of the $\tilde{\mathrm{T}}$ and all of the T containing a $2m+1-i$, taking into account the conditions (2), (3), and (4). The map preserves the highest weights of the representations corresponding to these tableaux in the respective decompositions $V(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0)$ and $V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0)$ using Littelmann's theorem, Theorem 4.1. For $\tilde{\mathrm{T}} \mapsto \mathrm{T}$, $c_{\mathrm{T}}(j) = c_{\tilde{\mathrm{T}}}(j)$ for all $j \neq 2m+1-i$, and $c_{\mathrm{T}}(2m+1-i) = c_{\tilde{\mathrm{T}}}(2m+1-i)+1$. Therefore,

$$\begin{aligned} &V(x_1 + c_{\tilde{\mathrm{T}}}(1) - c_{\tilde{\mathrm{T}}}(2m), \ldots, x_i - 1 + c_{\tilde{\mathrm{T}}}(i) - c_{\tilde{\mathrm{T}}}(2m+1-i), \ldots, \\ &\qquad x_k + c_{\tilde{\mathrm{T}}}(k) - c_{\tilde{\mathrm{T}}}(2m+1-k), c_{\tilde{\mathrm{T}}}(k+1), 0, \ldots, 0) \\ &= V(x_1 + c_{\mathrm{T}}(1) - c_{\mathrm{T}}(2m), \ldots, x_i + c_{\mathrm{T}}(i) - c_{\mathrm{T}}(2m+1-i), \ldots, \\ &\qquad x_k + c_{\mathrm{T}}(k) - c_{\mathrm{T}}(2m+1-k), c_{\mathrm{T}}(k+1), 0, \ldots, 0). \end{aligned}$$

For $x_i \neq x_{i+1}$, when $i = 1$, the representation

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m) \\ &\quad := V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0) \\ &\qquad - V(x_1 - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0) \end{aligned}$$

and when $2 \leq i \leq k-1$, the representation

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+1-i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+2-i) \\ &\qquad - F(V(x_1, \ldots, x_i - 1, \ldots, x_k, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0), 2m+2-i) \end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain any integer from the set $\{2m+1-i, \ldots, 2m\}$.

For $x_i = x_{i+1}$, when $i = 1$, the representation

$$F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m) := V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0)$$

and when $2 \leq i \leq k-1$, the representation

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+1-i) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+2-i) \end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain any integer from the set $\{2m+1-i, \ldots, 2m\}$.

For $i = k$, it is only important that $x_k \geq z$, which is true for any highest weight. The representation

$$\begin{aligned} &F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+1-k) \\ &\quad := F(V(x_1, \ldots, x_k, 0, \ldots, 0) \otimes V(z, 0, \ldots, 0), 2m+2-k) \\ &\qquad - F(V(x_1, \ldots, x_k - 1, 0, \ldots, 0) \otimes V(z-1, 0, \ldots, 0), 2m+2-k) \end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1, \ldots, x_k, 0, \ldots, 0) + v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z, 0, \ldots, 0)$ that are $(x_1, \ldots, x_k, 0, \ldots, 0)$-dominant and do not contain any integer from the set $\{2m+1-k, \ldots, 2m\}$.

The standard Young tableaux of shape $p(z,0,\dots,0)$ that are $(x_1,\dots,x_k,0,\dots,0)$-dominant and do not contain any integer from the set $\{2m+1-k,\dots,2m\}$ will only contain integers from the set $\{1,\dots,k+1\}$. Note that if T does not contain an integer i, this is the same as saying $c_{\mathrm{T}}(i)=0$.

For $x_i \neq x_{i-1}$, define a map from the standard Young tableaux, $\tilde{\mathrm{T}}$, of shape $p(z-1,0,\dots,0)$ that are $(x_1,\dots,x_i+1,\dots,x_k,0,\dots,0)$-dominant and only contain integers from the set $\{1,\dots,i,k+1\}$ to the standard Young tableaux, T, of shape $p(z,0,\dots,0)$ that are $(x_1,\dots,x_k,0,\dots,0)$-dominant and only contain integers from the set $\{1,\dots,i,k+1\}$ by sending $\tilde{\mathrm{T}}$ to the tableau formed by adding an i to the column.

This map is a bijection between all of the $\tilde{\mathrm{T}}$ and all of the T containing an i, taking into account the conditions (2), (3), and (4). The map preserves the highest weights of the representations corresponding to these tableaux in the respective decompositions $V(x_1,\dots,x_i+1,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0)$ and $V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0)$ using Littelmann's theorem, Theorem 4.1. For $\tilde{\mathrm{T}}\mapsto \mathrm{T}$, $c_{\mathrm{T}}(j)=c_{\tilde{\mathrm{T}}}(j)$ for all $j\neq i$, and $c_{\mathrm{T}}(i)=c_{\tilde{\mathrm{T}}}(i)+1$. Therefore,

$$\begin{aligned}&V(x_1+c_{\tilde{\mathrm{T}}}(1)-c_{\tilde{\mathrm{T}}}(2m),\dots,x_i+1+c_{\tilde{\mathrm{T}}}(i)-c_{\tilde{\mathrm{T}}}(2m+1-i),\dots,\\&\quad x_k+c_{\tilde{\mathrm{T}}}(k)-c_{\tilde{\mathrm{T}}}(2m+1-k),c_{\tilde{\mathrm{T}}}(k+1),0,\dots,0)\\&=V(x_1+c_{\mathrm{T}}(1)-c_{\mathrm{T}}(2m),\dots,x_i+c_{\mathrm{T}}(i)-c_{\mathrm{T}}(2m+1-i),\dots,\\&\qquad x_k+c_{\mathrm{T}}(k)-c_{\mathrm{T}}(2m+1-k),c_{\mathrm{T}}(k+1),0,\dots,0).\end{aligned}$$

For $x_i\neq x_{i-1}$, when $i=k$, the representation

$$\begin{aligned}&F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),k)\\&\quad:=V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0)\\&\qquad-V(x_1,\dots,x_k+1,0,\dots,0)\otimes V(z-1,0,\dots,0)\end{aligned}$$

and when $2\le i\le k-1$, the representation

$$\begin{aligned}&F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i)\\&\quad:=F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i+1)\\&\qquad-F(V(x_1,\dots,x_i+1,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0),i+1)\end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1,\dots,x_k,0,\dots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\dots,0)$ that are $(x_1,\dots,x_k,0,\dots,0)$-dominant and only contain integers from the set $\{1,\dots,i-1,k+1\}$.

For $x_i=x_{i-1}$, when $i=k$, the representation

$$\begin{aligned}&F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),k)\\&\quad:=F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2m+1-k)\end{aligned}$$

and when $2\le i\le k-1$, the representation

$$\begin{aligned}&F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i)\\&\quad:=F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),i+1)\end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1,\dots,x_k,0,\dots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\dots,0)$ that are $(x_1,\dots,x_k,0,\dots,0)$-dominant and only contain integers from the set $\{1,\dots,i-1,k+1\}$.

For $i=1$, there are no restrictions on the number of times 1 appears in a tableau (other than the size of the tableau), the representation

$$\begin{aligned}&F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),1)\\&\quad:=F(V(x_1,\dots,x_k,0,\dots,0)\otimes V(z,0,\dots,0),2)\\&\qquad-F(V(x_1+1,x_2,\dots,x_k,0,\dots,0)\otimes V(z-1,0,\dots,0),2)\end{aligned}$$

is equivalent to $\bigoplus_{\mathrm{T}} V((x_1,\ldots,x_k,0,\ldots,0)+v(\mathrm{T}))$ for all standard Young tableaux of shape $p(z,0,\ldots,0)$ that are $(x_1,\ldots,x_k,0,\ldots,0)$-dominant and only contain integers from the set $\{k+1\}$. The only tableau satisfying these conditions is the single column containing only $k+1$s. This tableau corresponds to the representation $V(x_1,\ldots,x_k,z,0,\ldots,0)$. Therefore,

$$V(x_1,\ldots,x_k,z,0,\ldots,0) = F(V(x_1,\ldots,x_k,0,\ldots,0)\otimes V(z,0,\ldots,0),1),$$

which is an integral combination of representations of the form $V(y_1,\ldots,y_k,0,\ldots,0)\otimes V(z-i,0,\ldots,0)$ for some y_j and i. Substituting in the integral combinations for all of the $V(y_1,\ldots,y_k,0,\ldots,0)$ yields the integral combination of $V(x_1,\ldots,x_k,z,0,\ldots,0) = V(x_1,\ldots,x_k,x_{k+1},0,\ldots,0)$ in the form of (1).

This algorithm can also be used when $z < 2k$. It can be applied until the size of z is exhausted, thus simplifying the problem of determining the integral combination to a reduced number of tableaux. If there are some equal terms, $x_i = x_{i+1}$, the algorithm may be completed for some $z < 2k$.

This algorithm also produces the following formula.

Proposition 5.1. *For any irreducible representation of $\mathfrak{sp}(2m,\mathbb{C})$ with highest weight $(x_1,\ldots,x_k,z,0,\ldots,0)$, such that $x_i \geq x_{i+1}+2$ when $1\leq i\leq k-1$ and $z\geq 2k$,*

$$\begin{aligned} &V(x_1,\ldots,x_k,z,0,\ldots,0)\\ &=\sum_{\substack{i_1,\ldots,i_k\in\{0,1\}\\ j_1,\ldots,j_k\in\{0,1\}}}(-1)^{i+j}\big(V(x_1-i_1+j_1,x_2-i_2+j_2,\ldots,x_k-i_k+j_k,0,\ldots,0)\\ &\qquad\qquad\otimes V(z-i-j,0,\ldots,0)\big)\end{aligned}$$

for $i=i_1+\ldots i_k$ and $j=j_1+\ldots j_k$.

6 Examples

For any irreducible representation of $\mathfrak{sp}(4,\mathbb{C})$ and $V=V(1,0)$, its formal combination is determined by

$$\begin{aligned} V(x,0) &= \mathrm{Sym}^x V && x\geq 0\\ V(x,1) &= \mathrm{Sym}^x V\otimes V - \mathrm{Sym}^{x+1}V - \mathrm{Sym}^{x-1}V && x\geq 1\\ V(x,y) &= \mathrm{Sym}^x V\otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V\otimes \mathrm{Sym}^{y-2}V && x\geq y\geq 2\\ &\quad - \mathrm{Sym}^{x-1}V\otimes \mathrm{Sym}^{y-1}V - \mathrm{Sym}^{x+1}V\otimes \mathrm{Sym}^{y-1}V. \end{aligned}$$

To apply the refinement of the recursive algorithm to the case of $\mathfrak{sp}(4,\mathbb{C})$ and some $V(x,y)$ such that $y\geq 2$, we do the following.

$$F(V(x,0)\otimes V(y,0),4) = V(x,0)\otimes V(y,0) - V(x-1,0)\otimes V(y-1,0).$$

$$F(V(x,0)\otimes V(y,0),1) = F(V(x,0)\otimes V(y,0),4) - F(V(x+1,0)\otimes V(y-1,0),4)$$

and

$$F(V(x+1,0)\otimes V(y-1,0),4) = V(x+1,0)\otimes V(y-1,0) - V(x,0)\otimes V(y-2,0).$$

Therefore,

$$\begin{aligned} &F(V(x,0)\otimes V(y,0),1)\\ &= V(x,0)\otimes V(y,0) - V(x-1,0)\otimes V(y-1,0)\\ &\qquad - (V(x+1,0)\otimes V(y-1,0) - V(x,0)\otimes V(y-2,0))\\ &= V(x,0)\otimes V(y,0) - V(x-1,0)\otimes V(y-1,0)\\ &\qquad - V(x+1,0)\otimes V(y-1,0) + V(x,0)\otimes V(y-2,0). \end{aligned}$$

and

$$\begin{aligned}
&V(x,y)\\
&= F(V(x,0)\otimes V(y,0),1)\\
&= V(x,0)\otimes V(y,0) - V(x-1,0)\otimes V(y-1,0)\\
&\quad - V(x+1,0)\otimes V(y-1,0) + V(x,0)\otimes V(y-2,0).
\end{aligned}$$

Equivalently,

$$V(x,y) = \sum_{i,j\in\{0,1\}} (-1)^{i+j} V(x-i+j,0)\otimes V(y-i-j,0).$$

For any irreducible representation of $\mathfrak{sp}(6,\mathbb{C})$ with highest weight $(x,y,0)$, its formal combination is determined similarly as above. For $V(x,y,z)$ such that $x \geq y+2$ and $z \geq 4$, the refinement to the recursive algorithm produces the following output:

$$\begin{aligned}
V(x,y,z) = {} & V(x,y,0)\otimes V(z,0,0) && - V(x-1,y,0)\otimes V(z-1,0,0)\\
& - V(x,y-1,0)\otimes V(z-1,0,0) && + V(x-1,y-1,0)\otimes V(z-2,0,0)\\
& - V(x,y+1,0)\otimes V(z-1,0,0) && + V(x-1,y+1,0)\otimes V(z-2,0,0)\\
& + V(x,y,0)\otimes V(z-2,0,0) && - V(x-1,y,0)\otimes V(z-3,0,0)\\
& - V(x+1,y,0)\otimes V(z-1,0,0) && + V(x,y,0)\otimes V(z-2,0,0)\\
& + V(x+1,y-1,0)\otimes V(z-2,0,0) && - V(x,y-1,0)\otimes V(z-3,0,0)\\
& + V(x+1,y+1,0)\otimes V(z-2,0,0) && - V(x,y+1,0)\otimes V(z-3,0,0)\\
& - V(x+1,y,0)\otimes V(z-3,0,0) && + V(x,y,0)\otimes V(z-4,0,0).
\end{aligned}$$

This is equivalent to

$$V(x,y,z) = \sum_{\substack{i_1,i_2\in\{0,1\}\\ j_1,j_2\in\{0,1\}}} (-1)^{i+j} V(x-i_1+j_1, y-i_2+j_2, 0)\otimes V(z-i-j,0,0)$$

where $i = i_1 + i_2$ and $j = j_1 + j_2$.

Substituting in for the irreducible representations with highest weights $(x', y', 0)$ and simplifying, this becomes

$$V(x,y,z) = \sum_{\substack{l_1,l_2,l_3\in\{0,\pm1,\pm2\}\\ \{|l_1|,|l_2|,|l_3|\}=\{0,1,2\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & 2\\ |l_1| & |l_2| & |l_3| \end{pmatrix} V(x+l_1,0,0)\otimes V(y+l_2-1,0,0) \otimes V(z+l_3-2,0,0).$$

7 A general formula

Now, we will expand the formula explicitly calculated in Section 6 for $V(x,y,z)$, such that $x \geq y+2$ and $z \geq 4$, to a general case for representations in $\mathfrak{sp}(2m,\mathbb{C})$ with highest weight vectors of sufficient size.

Theorem 7.1. *For any irreducible representation of $\mathfrak{sp}(2m,\mathbb{C})$, $V(x_1,\ldots,x_k,0,\ldots,0)$, such that $x_i \geq x_{i+1} + 2(k-1-i)$ when $1 \leq i \leq k-1$ and $x_k \geq 2k-2$,*

$$\begin{aligned}
&V(x_1,\ldots,x_k,0,\ldots,0)\\
&= \sum_{\substack{l_1,\ldots,l_k\in\{0,\pm1,\ldots,\pm(k-1)\}\\ \{|l_1|,\ldots,|l_k|\}=\{0,1,\ldots,k-1\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1\\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \bigotimes_{n=1}^{k} V(x_n + l_n - (n-1), 0, \ldots, 0).
\end{aligned}$$

Proof. We will argue by induction on k. The case when $k = 1$ is trivial. When $k = 2$ and $x_2 \geq 2$,

$$\begin{aligned} &V(x_1, x_2, 0, \ldots, 0) \\ &= \sum_{i,j\in\{0,1\}} (-1)^{i+j} V(x_1 - i + j, 0, \ldots, 0) \otimes V(x_2 - i - j, 0, \ldots, 0) \\ &= \sum_{\substack{l_1,l_2,\in\{0,\pm1\} \\ \{|l_1|,|l_2|\}=\{0,1\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 \\ |l_1| & |l_2| \end{pmatrix} V(x_1 + l_1, 0, \ldots, 0) \otimes V(x_2 + l_2 - 1, 0, \ldots, 0). \end{aligned}$$

Assume the statement of the theorem for k. Let $x_{k+1} = z$, we want to show

$$\begin{aligned} &V(x_1, \ldots, x_k, z, 0, \ldots, 0) \\ &= \sum_{\substack{l'_1,\ldots,l'_{k+1}\in\{0,\pm1,\ldots,\pm(k)\} \\ \{|l'_1|,\ldots,|l'_{k+1}|\}=\{0,1,\ldots,k\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k \\ |l'_1| & |l'_2| & \ldots & |l'_{k+1}| \end{pmatrix} \bigotimes_{n=1}^{k+1} V(x_n + l'_n - (n-1), 0, \ldots, 0) \end{aligned}$$

for $x_i \geq x_{i+1} + 2(k-i)$ when $1 \leq i \leq k$ and $z \geq 2k$. Call this sum S'. The tensor products in this sum are indexed by a k-tuple, $(l'_1, \ldots, l'_{k+1})$.

From Proposition 5.1,

$$\begin{aligned} &V(x_1, \ldots, x_k, z, 0, \ldots, 0) \\ &= \sum_{\substack{i_1,\ldots,i_k\in\{0,1\} \\ j_1,\ldots,j_k\in\{0,1\}}} (-1)^{i+j} \big(V(x_1 - i_1 + j_1, x_2 - i_2 + j_2, \ldots, x_k - i_k + j_k, 0, \ldots, 0) \\ &\qquad\qquad \otimes V(z - i - j, 0, \ldots, 0)\big). \end{aligned}$$

Applying the inductive hypothesis to $V(x_1 - i_1 + j_1, \ldots, x_k - i_k + j_k, 0, \ldots, 0)$ yields the following,

$$\begin{aligned} &V(x_1, \ldots, x_k, z, 0, \ldots, 0) \\ &= \sum_{\substack{i_1,\ldots,i_k\in\{0,1\} \\ j_1,\ldots,j_k\in\{0,1\}}} \sum_{\substack{l_1,\ldots,l_k\in\{0,\pm1,\ldots,\pm(k-1)\} \\ \{|l_1|,\ldots,|l_k|\}=\{0,1,\ldots,k-1\}}} (-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \\ &\qquad \big(\bigotimes_{n=1}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0)\big) \otimes V(z - i - j, 0, \ldots, 0). \end{aligned}$$

Call this sum S. The tensor products in this sum are indexed by three k-tuples of the form $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$. For a given $(l'_1, \ldots, l'_{k+1})$, we will show that there is exactly one $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ such that, for $1 \leq n \leq k$,

$$x_n - i_n + j_n + l_n - (n-1) = x_n + l'_n - (n-1)$$

and

$$(-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 & k \\ |l'_1| & |l'_2| & \ldots & |l'_k| & |l'_{k+1}| \end{pmatrix}.$$

We will now give an explicit description of how to calculate this $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ from $(l'_1, \ldots, l'_{k+1})$, and with some thought, it is easy enough to see that this is the only way to choose the proper index.

For a particular $(l'_1, \ldots, l'_{k+1})$, choose $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ in the following way. If $l'_{k+1} = k$, then $i+j = 0$ and $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ is equal to $\begin{pmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ l'_1 & \ldots & l'_k \end{pmatrix}$. If $l'_{k+1} = -k$, then $i+j = 2k$ and $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ is equal to $\begin{pmatrix} 1 & \ldots & 1 \\ 1 & \ldots & 1 \\ l'_1 & \ldots & l'_k \end{pmatrix}$. If $|l'_s| = k$ for $s \neq k+1$, take $i_s = 1$, $j_s = 0$, and $l_s = -(k-1)$ if $l'_s = -k$ and take $i_s = 0$, $j_s = 1$, and $l_s = k-1$ if $l'_s = k$. Next consider $|l'_r| = k-1$ and if $r \neq k+1$ take $i_r = 1$, $j_r = 0$, and $l_r = -(k-2)$ if $l'_r = -(k-1)$ and take $i_r = 0$, $j_r = 1$, and $l_r = k-2$ if $l'_s = k-1$. Continue with this process until $|l'_{k+1}| = k-t$ for some $0 < t \leq k$. For the other entries in $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$, take $l_a = l'_a$, and take $i_a = 0$ and $j_a = 0$ if $l'_{k+1} = k-t > 0$ and take $i_a = 1$ and $j_a = 1$ if $l'_{k+1} = -(k-t) < 0$. Note that if $l'_{k+1} = 0$, all of the entries have already been determined by the earlier process. Now for a particular element in S' indexed by $(l'_1, \ldots, l'_{k+1})$, we have the same element appearing in S indexed by the corresponding $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ and with the same sign attached.

The symmetric group on k letters acts on the elements of S by permuting the columns of the index of an element, $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$. Each $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ that corresonds to an element in S' as described above is the result of a permutation applied to one of four types. These four types are indexed by the following.

1) $\begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & \pm 1 & \ldots & \pm(k-1) \end{pmatrix}$

2) $\begin{pmatrix} 0 & \ldots & 0 & i_n & \ldots & i_k \\ 0 & \ldots & 0 & j_n & \ldots & j_k \\ 0 & \ldots & \pm(n-2) & \pm(n-1) & \ldots & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k$. Note that when $n = 1$, either $i_1 = 0$ and $j_1 = 1$ or $i_1 = 1$ and $j_1 = 0$.

3) $\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 0 & \pm 1 & \ldots & \pm(k-1) \end{pmatrix}$

4) $\begin{pmatrix} 1 & \ldots & 1 & i_n & \ldots & i_k \\ 1 & \ldots & 1 & j_n & \ldots & j_k \\ 0 & \ldots & \pm(n-2) & \pm(n-1) & \ldots & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k$. When $n = 1$, this coincides with the second type for $n = 1$.

Now we will prove by induction on k that $S = S'$ by showing that $S = S_1 + S_2$, where S_1 is a subsum containing only those elements corresponding to elements in the sum S', in other words S_1 is equal to the sum of all of the elements indexed by permutations of the four types of indices listed above, and $S_2 = S - S_1 = 0$. The case where $k = 1$ was shown earlier. In this case every term in S corresponded to a term in S' and there was no cancellation, so that $S_2 = 0$ trivially. The case where $k = 2$ was also explicitly calculated. Assume $S = S_1 + S_2$ such that $S_1 = S'$ and $S_2 = S - S_1 = 0$ for $k-1$ and take $k > 2$. We want to show $S = S_1 + S_2$ such that $S_1 = S'$ and $S_2 = S - S_1 = 0$ for k.

Consider all $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ with a fixed $l_r = \pm(k-1)$ and fixed i_r and j_r. Consider the following subsum contained in S,

$$\begin{aligned}
&S(r, \frac{l_r}{|l_r|}, i_r, j_r) \\
&= \Big(\sum_{\substack{i_1,\ldots,\hat{i_r},\ldots,i_k \in \{0,1\} \\ j_1,\ldots,\hat{j_r},\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,\hat{l_r},\ldots,l_k \in \{0,\pm1,\ldots,\pm(k-2)\} \\ \{|l_1|,\ldots,|\hat{l_r}|,\ldots,|l_k|\}=\{0,1,\ldots,k-2\}}} (-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \\
&\qquad \bigotimes_{\substack{n=1 \\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0) \otimes V(z - (i+j), 0, \ldots, 0) \Big) \\
&\qquad \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0).
\end{aligned}$$

This sum is equal to

$$\begin{aligned}
&(-1)^{(k-r)}(-1)^{(i_r+j_r)} \Big(\sum_{\substack{i_1,\ldots,\hat{i_r},\ldots,i_k \in \{0,1\} \\ j_1,\ldots,\hat{j_r},\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,\hat{l_r},\ldots,l_k \in \{0,\pm1,\ldots,\pm(k-2)\} \\ \{|l_1|,\ldots,|\hat{l_r}|,\ldots,|l_k|\}=\{0,1,\ldots,k-2\}}} (-1)^{i+j-i_r-j_r} \\
&\operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & & \ldots & k-2 \\ |l_1| & |l_2| & \ldots & |\hat{l_r}| & \ldots & |l_k| \end{pmatrix} \bigotimes_{\substack{n=1 \\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0) \\
&\otimes V((z - i_r - j_r) - (i + j - i_r - j_r), 0, \ldots, 0) \Big) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0) \\
&= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0).
\end{aligned}$$

The sum R is equal to

$$\begin{aligned}
&\sum_{\substack{i_1,\ldots,\hat{i_r},\ldots,i_k \in \{0,1\} \\ j_1,\ldots,\hat{j_r},\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,\hat{l_r},\ldots,l_k \in \{0,\pm1,\ldots,\pm(k-2)\} \\ \{|l_1|,\ldots,|\hat{l_r}|,\ldots,|l_k|\}=\{0,1,\ldots,k-2\}}} (-1)^{i+j-i_r-j_r} \\
&\operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & & \ldots & k-2 \\ |l_1| & |l_2| & \ldots & |\hat{l_r}| & \ldots & |l_k| \end{pmatrix} \bigotimes_{\substack{n=1 \\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0) \\
&\otimes V((z - i_r - j_r) - (i + j - i_r - j_r), 0, \ldots, 0).
\end{aligned}$$

Apply the inductive hypothesis to R. By the inductive hypothesis for $k-1$, R is equal to $R_1 + R_2$ such that R_1 contains a sum of elements indexed by permutations (from the symmetric group on $\{1, \ldots, r-1, r+1, \ldots, k\}$) of the four special types, with the r-th column removed, and $R_2 = R - R_1 = 0$. Therefore, $R = R_1$.

Now

$$\begin{aligned}
&S(r, \frac{l_r}{|l_r|}, i_r, j_r) \\
&= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0) \\
&= (-1)^{(k-r)}(-1)^{(i_r+j_r)}(R_1) \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0) \\
&= \Big(\sum_{\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}} (-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \\
&\qquad \bigotimes_{\substack{n=1 \\ n\neq r}}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0) \otimes V(z - (i+j), 0, \ldots, 0) \Big) \\
&\qquad \otimes V(x_r - i_r + j_r + l_r - (r-1), 0, \ldots, 0)
\end{aligned}$$

for all $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ with the fixed r-th column and the rest of the matrix equal to a permutation (from the symmetric group on $\{1, \dots, r-1, r+1, \dots, k\}$) of one of the four types.

$$S = \sum_{\substack{r \in \{1,\dots,k\} \\ \epsilon \in \{-1,+1\} \\ i_r, j_r \in \{0,1\}}} S(r, \epsilon, i_r, j_r)$$

for the subsums $S(r, \epsilon, i_r, j_r)$, and all of these have been reduced by the inductive hypothesis.

We will show that for the remaining elements in the subsum $S_2 = S - S_1$, which have not been cancelled out by the application of the inductive hypothesis, there is a well-defined pairing of elements into disjoint pairs such that the sum of the elements in a pair is equal to zero. Notice that two indices $\begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ and $\begin{pmatrix} i'_1 & \dots & i'_k \\ j'_1 & \dots & j'_k \\ l'_1 & \dots & l'_k \end{pmatrix}$ will correspond to elements that will sum to zero if $-(i_1, \dots, i_k) + (j_1, \dots, j_k) + (l_1, \dots, l_k) = -(i'_1, \dots, i'_k) + (j'_1, \dots, j'_k) + (l'_1, \dots, l'_k)$, $i + j = i' + j'$, and the signs associated to $(l_1, \dots, l_k)$ and $(l'_1, \dots, l'_k)$ are different.

Define the set $\mathcal{M}$ to be the elements in the reduced $S_2 = S - S_1$. This means all of the elements indexed by a matrix $M = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ such that if the $l_r = \pm k - 1$, then M is not a permutation of one of the four special types of indices but with the r-th column removed it is a permutation of one of the four special types of indices (for $k-1$). This means any M is a permutation of one of the following.

1) $\begin{pmatrix} 0 & 0 & \dots & 0 & i_k \\ 0 & 0 & \dots & 0 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$. The last two columns are equal to one of the following $\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \pm(k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \pm(k-2) & -(k-1) \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ \pm(k-2) & \pm(k-1) \end{pmatrix}$.

2) $\begin{pmatrix} 0 & \dots & 0 & i_n & \dots & i_{k-1} & i_k \\ 0 & \dots & 0 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \leq n \leq k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \leq r \leq k-1$. The last two columns are equal to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -(k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -(k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -(k-2) & -(k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -(k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ (k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ (k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ (k-2) & -(k-1) \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ (k-2) & \pm(k-1) \end{pmatrix}$.

3) $\begin{pmatrix} 1 & 1 & \dots & 1 & i_k \\ 1 & 1 & \dots & 1 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$. The last two columns are equal to one of the following, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ \pm(k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \pm(k-2) & -(k-1) \end{pmatrix}$, or $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \pm(k-2) & \pm(k-1) \end{pmatrix}$.

4) $\begin{pmatrix} 1 & \dots & 1 & i_n & \dots & i_{k-1} & i_k \\ 1 & \dots & 1 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \le n \le k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \le r \le k-1$. The last two columns are equal to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -(k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -(k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -(k-2) & -(k-1) \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -(k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ (k-2) & \pm(k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ (k-2) & (k-1) \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ (k-2) & -(k-1) \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ (k-2) & \pm(k-1) \end{pmatrix}$.

Now define a function $\Xi : \mathcal{M} \to \mathcal{M}$. We will define it for elements with these four types of indices in terms of their indices. Then the definition for any other element can be found by endowing Ξ with the property that $\Xi(\sigma M) = \sigma\Xi(M)$ for any $\sigma \in S_k$ and any index M of an element in $\mathcal{M}$. Ξ is now defined for all elements in $\mathcal{M}$ because any index of an element can be found as a permutation of one of these four types of indices.

1) Given indices of the form $\begin{pmatrix} 0 & 0 & \dots & 0 & i_k \\ 0 & 0 & \dots & 0 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$, define Ξ in the following way.

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

2) Given indices of the form $\begin{pmatrix} 0 & \dots & 0 & i_n & \dots & i_{k-1} & i_k \\ 0 & \dots & 0 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \le n \le k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \le r \le k-1$, define the Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n = k - 1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n > k - 1$, either $i_{k-2} = 1$, $j_{k-2} = 0$, and $l_{k-2} = -(k-3)$ or $i_{k-2} = 0$, $j_{k-2} = 1$, and $l_{k-2} = k - 3$. Define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & -(k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & -(k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & (k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & (k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & (k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & (k-1) & \dots & -(k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & -(k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & -(k-1) & \dots & -(k-3) \end{pmatrix}$$

3) Given indices of the form $\begin{pmatrix} 1 & 1 & \dots & 1 & i_k \\ 1 & 1 & \dots & 1 & j_k \\ 0 & \pm 1 & \dots & \pm(k-2) & \pm(k-1) \end{pmatrix}$, define Ξ in the following way.

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 0 & 1 \\ 0 & \pm 1 & \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 0 & \pm 1 & \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 0 \\ 0 & \pm 1 & \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \\ 0 & \pm 1 & \dots & -(k-1) & -(k-2) \end{pmatrix}$$

4) Given indices of the form $\begin{pmatrix} 1 & \dots & 1 & i_n & \dots & i_{k-1} & i_k \\ 1 & \dots & 1 & j_n & \dots & j_{k-1} & j_k \\ 0 & \dots & \pm(n-2) & \pm(n-1) & \dots & \pm(k-1) & \pm(k-1) \end{pmatrix}$ for some $1 \le n \le k-1$ and with $i_r = 0, j_r = 1$ for $l_r = r-1$ and $i_r = 1, j_r = 0$ for $l_r = -(r-1)$ for $n \le r \le k-1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 0 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 0 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 0 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & 1 \\ \dots & 1 & 0 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n = k-1$, define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & 1 \\ \dots & 0 & 0 \\ \dots & -(k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & 0 \\ \dots & 0 & 1 \\ \dots & -(k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & -(k-1) & -(k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 1 \\ \dots & 1 & 0 \\ \dots & (k-2) & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & (k-1) & (k-2) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & 0 \\ \dots & 1 & 1 \\ \dots & (k-2) & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & 1 \\ \dots & 0 & 1 \\ \dots & (k-1) & -(k-2) \end{pmatrix}$$

When $n > k-1$, either $i_{k-2} = 1$, $j_{k-2} = 0$, and $l_{k-2} = -(k-3)$ or $i_{k-2} = 0$, $j_{k-2} = 1$, and $l_{k-2} = k-3$. Define Ξ in the following way.

$$\begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & -(k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & -(k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & (k-3) & \dots & (k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & (k-1) & \dots & (k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 0 & \dots & 0 \\ \dots & 1 & \dots & 1 \\ \dots & (k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 1 & \dots & 1 \\ \dots & 0 & \dots & 0 \\ \dots & (k-1) & \dots & -(k-3) \end{pmatrix}$$

$$\begin{pmatrix} \dots & 1 & \dots & 0 \\ \dots & 0 & \dots & 1 \\ \dots & -(k-3) & \dots & -(k-1) \end{pmatrix} \mapsto \begin{pmatrix} \dots & 0 & \dots & 1 \\ \dots & 1 & \dots & 0 \\ \dots & -(k-1) & \dots & -(k-3) \end{pmatrix}$$

Ξ is well-defined because it sends any element in $\mathcal{M}$ to another such element. This is obviously true for the elements with the particular indices Ξ was explicitly defined for, and since $\mathcal{M}$ is invariant under S_k, this is true for all elements in $\mathcal{M}$. It is also easy enough to verify that $\Xi^2 = Id$. Let $[M]$ be the element indexed by the matrix M, and let $\operatorname{sgn}(M)$ be the sign associated to that element. Then $S_2 = \sum_{M \in \mathcal{M}} [M]$. Also, $|\{M \in \mathcal{M} \,|\, \operatorname{sgn}(M) = 1\}| = |\{M \in \mathcal{M} \,|\, \operatorname{sgn}(M) = -1\}|$ and $\operatorname{sgn}(\Xi(M)) = -\operatorname{sgn}(M)$. Therefore Ξ is a bijection between $\{M \in \mathcal{M} \,|\, \operatorname{sgn}(M) = 1\}$ and $\{M \in \mathcal{M} \,|\, \operatorname{sgn}(M) = -1\}$.

$$S_2 = \sum_{\substack{M \in \mathcal{M} \\ \operatorname{sgn}(M)=1}} [M] + \sum_{\substack{M \in \mathcal{M} \\ \operatorname{sgn}(M)=1}} [\Xi(M)] = \sum_{\substack{M \in \mathcal{M} \\ \operatorname{sgn}(M)=1}} ([M] + [\Xi(M)])$$

We claim that $[M] + [\Xi(M)] = 0$ for every $M \in \mathcal{M}$. Again, we only need to consider M as one of the four special types because for any other index σM for some $\sigma \in S_k$ will have $[\sigma M] + [\Xi(\sigma M)] = [\sigma M] + [\sigma \Xi(M)] = \sigma([M] + [\Xi(M)]) = \sigma(0) = 0$.

To see $[M] + [\Xi(M)] = 0$ for some M that is one of the four special types, it is enough to show M and $\Xi(M)$ satisfy the three necessary conditions. For $M = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \\ l_1 & \dots & l_k \end{pmatrix}$ and $\Xi(M) = \begin{pmatrix} i'_1 & \dots & i'_k \\ j'_1 & \dots & j'_k \\ l'_1 & \dots & l'_k \end{pmatrix}$, it is an easy calculation to check

$$-(i_1, \dots, i_k) + (j_1, \dots, j_k) + (l_1, \dots, l_k) = -(i'_1, \dots, i'_k) + (j'_1, \dots, j'_k) + (l'_1, \dots, l'_k).$$

Also, $\Xi(M)$ does not change the number of entries equal to 1 in the i_r and j_s slots for M. Therefore $i + j = i' + j'$. Also, $\Xi(M)$ involves a transposition of $(l_1, \dots, l_k)$, so the signs associated to the elements are different.

Therefore, $S_2 = \sum_{\substack{M \in \mathcal{M} \\ \operatorname{sgn}(M)=1}} 0 = 0$. This completes the proof.

□

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