Research Statement

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1 Introduction

Finite-dimensional representations of classical Lie algebras have been studied since the 19th century, and have found many applications in Mathematics and Physics. It is therefore surprising that basic questions remain unanswered, even in low-dimensional cases. For example, an explicit formula for the weight multiplicities in finite-dimensional representations of $\mathfrak{sp}(4, \mathbb{C})$ has only been published in 2004, [3]. Many researchers are interested in classical Lie algebras for a myriad of reasons. In particular many questions and applications are related to the fundamental problem of finding weight multiplicities and determining the decomposition of a tensor product of irreducible representations into irreducible representations.

Along similar lines, my current research is focused on the representation theory of the rank m symplectic Lie algebra $\mathfrak{sp}(2m, \mathbb{C})$. One of my results proves an arbitrary irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation.

The original motivation for this line of research was to calculate L- and ε- factors for representations of the real Weil group, which requires precise multiplicity information. However, the results obtained are of independent interest beyond this immediate application.

In [1] and [9], the authors, Akin and Zelevinskii respectively, independently prove an identity expressing any irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ as a formal sum of tensor products of symmetric powers of the standard representation using resolutions. In [8], I use a different approach for Lie algebras of type C_m.

2 Main theorems

I have two main results. The first of which is Theorem 5.1. It states that any irreducible representation of $\mathfrak{sp}(2m, \mathbb{C})$ can be expressed as a formal sum of tensor products of symmetric powers of the standard representation. In the method of proof, there is an algorithm for finding such a formal sum using the highest weight of the representation and Young diagrams by implementing Littelmann's results from [7]. This algorithm can be refined and leads to Proposition 5.2, which is a first step toward calculating the formal sum for an irreducible representation with highest weight sufficiently large and away from the walls of the dominant cone.

My second main result is Theorem 5.3. It gives an explicit definition of the formal sum of tensor products of symmetric powers of the standard representation for an irreducible representation with highest weight suffieciently large and away from the walls of the dominant cone. It uses Proposition 5.2 but requires more work, and I have provided a sketch of the idea of the proof towards the end of this statement.

Before presenting these theorems, I will first explain my initial result, which stemmed from an elementary approach using multilinear algebra. For illustration I will describe the $\mathfrak{sp}(4, \mathbb{C})$ case, and then the general rank case.

3 An elementary approach using multilinear algebra

The Lie algebra $\mathfrak{sp}(2m, \mathbb{C})$ has the complete reducibility property and every irreducible representation can be indexed by a highest weight regarded as an m-tuple, $(x_1, \ldots, x_m)$, where $x_i \in \mathbb{Z}_{\geq 0}$ and $x_1 \geq x_2 \geq \ldots \geq x_m$. We will identify the irreducible representation with this highest weight as $V(x_1, \ldots, x_k)$. The most basic nontrivial irreducible representations of $\mathfrak{sp}(2m, \mathbb{C})$ is the standard representation, which is $V(1, 0, \ldots, 0)$. The standard representation has the nice property that $\mathrm{Sym}^n V(1, 0, \ldots, 0) = V(n, 0, \ldots, 0)$. Furthermore, the standard representation can be used as a building block for determining every irreducible representation. For example, given $V = V(1, 0, \ldots, 0)$ and for $x \geq y \geq 1$,

$$\mathrm{Sym}^x V \otimes \mathrm{Sym}^y V = (\bigoplus_{p=0}^{y} V(x + y - p, p, 0, \ldots, 0)) \oplus (\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V).$$

Therefore in the Grothendieck group of all representations of $\mathfrak{sp}(2m, \mathbb{C})$, for $V = V(1, 0, \ldots, 0)$, we get

$$\begin{aligned}
V(x, 0, \ldots, 0) &= \mathrm{Sym}^x V \\
V(x, 1, 0, \ldots, 0) &= \mathrm{Sym}^x V \otimes V - \mathrm{Sym}^{x+1} V - \mathrm{Sym}^{x-1} V \\
V(x, y, 0, \ldots, 0) &= \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V \otimes \mathrm{Sym}^{x-2} V \\
&\quad - \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V - \mathrm{Sym}^{x+1} V \otimes \mathrm{Sym}^{y-1} V.
\end{aligned}$$

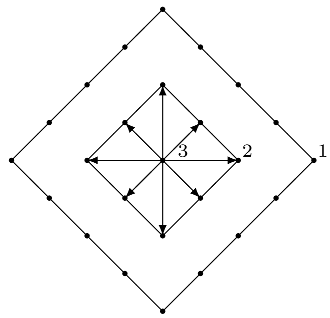
This can be shown using an elementary approach involving a result from multilinear algebra and an explicit determination of highest weight vectors. The result from multilinear algebra proves there is an injective intertwining map from $\mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} W$ to $\mathrm{Sym}^x V \otimes \mathrm{Sym}^y W$ if there is a trivial representation contained in $V \otimes W$. This result can be applied in this situation because the standard representation of $\mathfrak{sp}(2m, \mathbb{C})$ is self-dual.

4 The rank 2 case

Every representation of $\mathfrak{sp}(4, \mathbb{C})$ can now be expressed as a formal sum of tensor products of symmetric powers of the standard representation. For $V = V(1, 0)$ and $x \geq y \geq 2$,

$$\begin{aligned}
V(x, 0) &= \mathrm{Sym}^x V \\
V(x, 1) &= \mathrm{Sym}^x V \otimes V - \mathrm{Sym}^{x+1} V - \mathrm{Sym}^{x-1} V \\
V(x, y) &= \mathrm{Sym}^x V \otimes \mathrm{Sym}^y V + \mathrm{Sym}^x V \otimes \mathrm{Sym}^{y-2} V \\
&\quad - \mathrm{Sym}^{x+1} V \otimes \mathrm{Sym}^{y-1} V - \mathrm{Sym}^{x-1} V \otimes \mathrm{Sym}^{y-1} V
\end{aligned}$$

The weight diagram for $V(n, 0) = \mathrm{Sym}^n V(1, 0)$ is a series of nested diamonds with leading weights $(n - 2i, 0)$ and with multiplicities $i + 1$ along the diamonds, $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The following is the weight diagram for $V_{(4,0)}$.



By the previous corollary, the multiplicity of a weight of $V(x,y)$ can be calculated in an obvious way after finding the multiplicity of that weight in the appropriate tensor products $V(r,0)\otimes V(s,0)$.

5 The general rank case

This idea of using the standard representation as a building block for determining every irreducible representation can then be expanded to more complicated highest weights, but more machinery is needed. In [7], Littelmann provides a generalization of the Littlewood-Richardson rule in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. I am using Littelmann's work generalizing the Littlewood-Richardson rule to $\mathrm{Sp}(2m)$, type C_m. Littelmann's main theorem for $Sp(2m)$ utilizes Young diagrams to decompose a tensor product of two irreducible representations. Using Littelmann's theorem, I can prove the following result.

Theorem 5.1. *Any irreducible representation $V(x_1,\ldots,x_k,0,\ldots,0)$ of $\mathfrak{sp}(2m,\mathbb{C})$ can be written as an integral combination in the form*

$$\sum_{i=1}^{n} c_i V(y_1^{(i)},0,\ldots,0)\otimes\ldots\otimes V(y_k^{(i)},0,\ldots,0)$$

for some n, c_i, and $y_j^{(i)}$.

Note that $V(n,0,\ldots,0)=\mathrm{Sym}^n V(1,0,\ldots,0)$, and this statement could be replaced with the following,

$$V(x_1,\ldots,x_k,0,\ldots,0)=\sum_{i=1}^{n} c_i \mathrm{Sym}^{y_1^{(i)}} V\otimes\ldots\otimes\mathrm{Sym}^{y_k^{(i)}} V$$

for $V=V(1,0,\ldots,0)$ and some $n\in\mathbb{N}$, $c_i\in\mathbb{Z}$, and $y_j^{(i)}\in\mathbb{Z}_{\geq 0}$.

Littelmann's theorem and a combinatorics argument also yield a recursive algorithm for determining the formal combination in most cases. It is still a work in progress to find precise formulas for every representation, but I have proven two formulas for general cases. The following formula is the first step in calculating the formal sum of tensor products of symmetric powers of the standard representation for a representation with a highest weight that satisfies certain conditions.

Proposition 5.2. *For any irreducible representation of* $\mathfrak{sp}(2m,\mathbb{C})$*,* $V(x_1,\ldots,x_k,z,0,\ldots,0)$*, such that* $x_i \geq x_{i+1}+2$ *when* $1 \leq i \leq k-1$ *and* $z \geq 2k$*,*

$$\begin{aligned} &V(x_1,\ldots,x_k,z,0,\ldots,0)\\ &= \sum_{\substack{i_1,\ldots,i_k\in\{0,1\}\\ j_1,\ldots,j_k\in\{0,1\}}} (-1)^{i+j}\big(V(x_1-i_1+j_1,x_2-i_2+j_2,\ldots,x_k-i_k+j_k,0,\ldots,0)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\otimes V(z-i-j,0,\ldots,0)\big) \end{aligned}$$

for $i = i_1+\ldots i_k$ *and* $j = j_1+\ldots j_k$*.*

Proposition 5.2 falls out directly using combinatorics and a refinement of the recursive algorithm mentioned above. This formula can be seen in the following example, which was shown at the end of Section 3. For $x \geq y \geq 2$ and $V = V(1,0,\ldots,0)$,

$$\begin{aligned} &V(x,y,0,\ldots,0)\\ &= \sum_{i,j\in\{0,1\}} (-1)^{i+j} V(x-i+j,0,\ldots,0)\otimes V(y-i-j,0,\ldots,0)\\ &= \sum_{i,j\in\{0,1\}} (-1)^{i+j}\mathrm{Sym}^{x-i+j}V\otimes \mathrm{Sym}^{y-i-j}V\\ &= \mathrm{Sym}^{x}V\otimes\mathrm{Sym}^{y}V+\mathrm{Sym}^{x}V\otimes\mathrm{Sym}^{y-2}V\\ &\quad -\mathrm{Sym}^{x+1}V\otimes\mathrm{Sym}^{y-1}V-\mathrm{Sym}^{x-1}V\otimes\mathrm{Sym}^{y-1}V. \end{aligned}$$

The next formula gives the exact formal sum of tensor products of symmetric powers of the standard representation for a representation with a highest weight that satisfies certain conditions.

Theorem 5.3. *For any irreducible representation of* $\mathfrak{sp}(2m,\mathbb{C})$*,* $V(x_1,\ldots,x_k,0,\ldots,0)$*, such that* $x_i \geq x_{i+1}+2(k-1-i)$ *when* $1\leq i\leq k-1$ *and* $x_k \geq 2k-2$*,*

$$\begin{aligned} &V(x_1,\ldots,x_k,0,\ldots,0)\\ &= \sum_{\substack{l_1,\ldots,l_k\in\{0,\pm1,\ldots,\pm(k-1)\}\\ \{|l_1|,\ldots,|l_k|\}=\{0,1,\ldots,k-1\}}} \mathrm{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1\\ |l_1| & |l_2| & \ldots & |l_k|\end{pmatrix}\bigotimes_{n=1}^{k} V(x_n+l_n-(n-1),0,\ldots,0). \end{aligned}$$

Strinkingly, all of coefficients are ±1 here. In general, this is not always the case. For example, in $\mathfrak{sp}(6,\mathbb{C})$,

$$V(1,1,1) = V(1,0,0)\otimes V(1,0,0)\otimes V(1,0,0) - 2V(2,0,0)\otimes V(1,0,0) + V(3,0,0) - V(1,0,0).$$

Theorem 5.3 is the result of Propositon 5.2 along with a combinatorics argument. This formula can be seen in the following example, which was shown at the end of Section 3. For $x \geq y \geq 2$ and $V = V(1,0,\ldots,0)$

$$\begin{aligned} &V(x,y,0,\ldots,0)\\ &= \sum_{\substack{l_1,l_2,\in\{0,\pm1\}\\ \{|l_1|,|l_2|\}=\{0,1\}}} \mathrm{sgn}\begin{pmatrix} 0 & 1\\ |l_1| & |l_2|\end{pmatrix} V(x+l_1,0,\ldots,0)\otimes V(y+l_2-1,0,\ldots,0)\\ &= \sum_{\substack{l_1,l_2,\in\{0,\pm1\}\\ \{|l_1|,|l_2|\}=\{0,1\}}} \mathrm{sgn}\begin{pmatrix} 0 & 1\\ |l_1| & |l_2|\end{pmatrix} \mathrm{Sym}^{x+l_1}V\otimes\mathrm{Sym}^{y+l_2-1}V\\ &= \mathrm{Sym}^{x}V\otimes\mathrm{Sym}^{y}V+\mathrm{Sym}^{x}V\otimes\mathrm{Sym}^{y-2}V\\ &\quad -\mathrm{Sym}^{x+1}V\otimes\mathrm{Sym}^{y-1}V-\mathrm{Sym}^{x-1}V\otimes\mathrm{Sym}^{y-1}V. \end{aligned}$$

I will now present a sketch of the method used to prove Theorem 5.3. The details of the proof are rather involved and are contained in [8]. The idea of the proof is to use induction on k and in the induction step to use Proposition 5.2. Letting $x_{k+1} = z$, I want to show

$$V(x_1, \ldots, x_k, z, 0, \ldots, 0) = \sum_{\substack{l'_1,\ldots,l'_{k+1} \in \{0,\pm 1,\ldots,\pm(k)\} \\ \{|l'_1|,\ldots,|l'_{k+1}|\} = \{0,1,\ldots,k\}}} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k \\ |l'_1| & |l'_2| & \ldots & |l'_{k+1}| \end{pmatrix} \bigotimes_{n=1}^{k+1} V(x_n + l'_n - (n-1), 0, \ldots, 0)$$

for $x_i \geq x_{i+1} + 2(k-i)$ when $1 \leq i \leq k$ and $z \geq 2k$. Call this sum S'. The tensor products in this sum are indexed by a k-tuple, $(l'_1, \ldots, l'_{k+1})$.

From Proposition 5.2,

$$V(x_1, \ldots, x_k, z, 0, \ldots, 0) = \sum_{\substack{i_1,\ldots,i_k \in \{0,1\} \\ j_1,\ldots,j_k \in \{0,1\}}} (-1)^{i+j} (V(x_1 - i_1 + j_1, x_2 - i_2 + j_2, \ldots, x_k - i_k + j_k, 0, \ldots, 0) \otimes V(z - i - j, 0, \ldots, 0)).$$

Applying the inductive hypothesis to $V(x_1 - i_1 + j_1, \ldots, x_k - i_k + j_k, 0, \ldots, 0)$ yields the following,

$$V(x_1, \ldots, x_k, z, 0, \ldots, 0) = \sum_{\substack{i_1,\ldots,i_k \in \{0,1\} \\ j_1,\ldots,j_k \in \{0,1\}}} \sum_{\substack{l_1,\ldots,l_k \in \{0,\pm 1,\ldots,\pm(k-1)\} \\ \{|l_1|,\ldots,|l_k|\} = \{0,1,\ldots,k-1\}}} (-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} \left(\bigotimes_{n=1}^{k} V(x_n - i_n + j_n + l_n - (n-1), 0, \ldots, 0)\right) \otimes V(z - i - j, 0, \ldots, 0).$$

Call this sum S. The tensor products in this sum are indexed by three k-tuples of the form $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$. For a given $(l'_1, \ldots, l'_{k+1})$, there is exactly one $\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \\ l_1 & \ldots & l_k \end{pmatrix}$ such that, for $1 \leq n \leq k$,

$$x_n - i_n + j_n + l_n - (n-1) = x_n + l'_n - (n-1)$$

and

$$(-1)^{i+j} \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 \\ |l_1| & |l_2| & \ldots & |l_k| \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} 0 & 1 & \ldots & k-1 & k \\ |l'_1| & |l'_2| & \ldots & |l'_k| & |l'_{k+1}| \end{pmatrix}.$$

Therefore, there is a subsum, S_1, in S such that $S_1 = S'$ by summing the elements with the appropriate indices.

I then prove by induction on k that $S = S'$ by showing that $S = S_1 + S_2$ and $S_2 = S - S_1 = 0$. The case where $k = 1$ was shown earlier. In this case every term in S corresponded to a term in S' and there was no cancellation, so that $S_2 = 0$ trivially. The inductive step uses the inductive hypothesis to reduce S_2, and then to show the reduced version of $S_2 = 0$ requires a smart pairing of the remaining elements such that the sum of two elements in each pair equals to zero.

6 Future research

My plans for the future involve expanding the research I've already done to include more cases and using my research for applications.

- I have found a formula for a general case of representations of a symplectic Lie algebra with highest weights satisfying certain properties, but there are still boundary cases to consider. Not every case contains a formal sum with coefficients of just ± 1. For example, in $\mathfrak{sp}(6, \mathbb{C})$,

  $$V(1,1,1) = V(1,0,0) \otimes V(1,0,0) \otimes V(1,0,0) - 2V(2,0,0) \otimes V(1,0,0) + V(3,0,0) - V(1,0,0).$$

  I will attempt to determine formulas for the limit cases when expressing any representation as a formal sum of tensor products of symmetric powers of the standard representation for symplectic Lie algebras and to understand of what significance is it if a sum has some coefficient not equal to ± 1.

- The orginal motivation for this area of research was to determine L- and ε- factors for representations of the real Weil group. Representations of the real Weil group to $\mathfrak{sp}(2m, \mathbb{C})$ can be composed with representations of $\mathfrak{sp}(2m, \mathbb{C})$. Viewing a representation as a formal sum of tensor products of symmetric powers of the standard representations reduces the problem of calculating these factors the just the case of a tensor product of symmetric powers of the standard representation, which is a more easily understood object. I plan to exploit the formula from Theorem 5.3 to explicitly calculate L- and ε-factors for symplectic L-parameters in any degre.

- In 2004, Cagliero and Tirao wrote a paper, [3], providing a closed formula for the weight multiplicities in any representation of $\mathfrak{sp}(4, \mathbb{C})$. I hope to apply the results I have already found to formulate this multiplicity formula in a more elementary way using the formal sums I have found. After that, I hope to formulate other multiplicity formulas for higher rank symplectic Lie algebras using similar methods.

- In [7], Littelmann used Young tableaux to decompose a tensor product of irreducible representations in the cases of all simple, simply connected algebraic groups of type A_m, B_m, C_m, D_m, G_2, E_6, and partial results for F_4, E_7, and E_8. I plan on trying to use ideas similar to what I did in the C_m case to produce analogous results in the other cases. Since B_2 is equivalent to C_2, I am hopeful that there will be similar results for B_m. The A_m case has already been worked out in [1] and [9].

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