Week 7 worksheet solutions

1. Consider the vector space $\mathbb{R}^2$ endowed with the inner product given by

$$\langle \begin{bmatrix} a \\ b \\ \end{bmatrix} , \begin{bmatrix} c \\ d \\ \end{bmatrix} \rangle = ad - bc + 5bd.$$

(a) Find the lengths of $\begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}$.

$$\| \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} \| = \left( \langle \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} \rangle \right)^{1/2} = \sqrt{(1)(1) - (1)(0) - (0)(1) + 5(0)(0)} = 1$$

$$\| \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} \| = \left( \langle \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} \rangle \right)^{1/2} = \sqrt{(0)(0) - (0)(1) - (1)(0) + 5(1)(1)} = \sqrt{5}$$

(b) Find the cosine of the angle between $\begin{bmatrix} 1 \\ 0 \\ \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}$.

$$\cos \theta = \frac{\langle \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} \rangle}{\langle \begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} \rangle^{1/2} \langle \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} \rangle^{1/2}} = -\frac{1}{\sqrt{5}}$$

2. Consider the vector space $C[0,1]$ of functions continuous between zero and one, endowed with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt$.

(a) Find the length of $f(t) = e^t$.

$$\| e^t \| = (e^t, e^t)^{1/2} = \int_0^1 e^{2t} \, dt = \sqrt{\frac{1}{2}(e^{2t}|_{t=0}^1)} = \sqrt{\frac{1}{2}(e^2 - 1)}$$

(b) Find all values for $a$ and $b$ so that $g(t) = at + b$ is orthogonal to $f(t) = t + 1$.

$at + b$ and $t + 1$ are orthogonal precisely if $\langle at + b, t + 1 \rangle = 0$.

Thus we solve

$$0 = \int_0^1 (at + b)(t + 1) \, dt = \int_0^1 (at^2 + (a + b)t + b) \, dt = \frac{a}{3} + \frac{a + b}{2} + b.$$

Thus in order for $at+b$ to be orthogonal to $t+1$, we need that

$$\frac{5a}{6} + \frac{3b}{2} = 0 \text{ or } 5a + 9b = 0.$$

So any values of $a$ and $b$ where $b = -\frac{5}{9}a$ will work.
3. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by \[
\begin{bmatrix}
2 \\
0 \\
-1 \\
3
\end{bmatrix},
\begin{bmatrix}
-6 \\
1 \\
5 \\
-8
\end{bmatrix}.
\] Find a basis for $W^\perp$.

We are looking for a vector in $\mathbb{R}^4$ that dots with each of the two given vectors to give zero. Thus we solve

\[
2a + 0b - 1c + 3d = 0 \quad \text{and} \quad -6a + 1b + 5c - 8d = 0.
\]

This turns into the following augmented matrix, which row reduces as shown:

\[
\begin{bmatrix}
2 & 0 & -1 & 3 & | & 0 \\
-6 & 1 & 5 & -8 & | & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
2 & 0 & -1 & 3 & | & 0 \\
0 & 1 & 2 & 1 & | & 0
\end{bmatrix}.
\]

A general solution to this system has the form

\[
\begin{bmatrix}
(s - 3r)/2 \\
-2s - r \\
s \\
r
\end{bmatrix} =
\begin{bmatrix}
-3/2 \\
-1 \\
0 \\
1
\end{bmatrix} +
\begin{bmatrix}
1/2 \\
-2 \\
1 \\
0
\end{bmatrix}.
\]

Thus these two vectors span $W^\perp$.

4. In $C[0,1]$, find the projection of $t + 1$ onto $t^2$.

\[
\text{proj}_{t^2}(t + 1) = \frac{\langle t + 1, t^2 \rangle}{\langle t^2, t^2 \rangle} t^2 = \frac{\int_0^1 (t^3 + t^2) \, dt}{\int_0^1 t^4 \, dt} t^2 = \frac{7/12}{1/5} t^2 = \frac{7}{60} t^2.
\]

5. Let $W$ be the subspace of $\mathbb{R}^3$ spanned by \[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}\] and \[
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}.
\]

(a) Find the distance between \[
\begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\] and the nearest vector in $W$.

Let $v$ denote the given vector, and let $w_1$ and $w_2$ denote the two given vectors in $W$. Note that $w_1$ and $w_2$ are orthogonal. The nearest vector in $W$ to $v$ is the projection of $v$ onto $W$, which is given by

\[
\text{proj}_W v = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \frac{4}{2} \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} + \frac{2}{1} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
2 \\
2 \\
2
\end{bmatrix}.
\]
The distance between $v$ and $W$ is just the distance between $v$ and this projection. This in turn is the length of the difference of the two, which is given by

$$\| \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \| = \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle^{1/2} = \sqrt{1 + 1} = \sqrt{2}.$$

(b) Write $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ as $w + u$, where $w$ is in $W$ and $u$ is in $W^\perp$.

We’ve already found the projection of $v$ onto $W$, so we set $w$ to be this projection. Then $u$ is just the difference between them (which is automatically in $W^\perp$). Thus we have

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$