Week 3 worksheet solutions

1. Let $L : P_1 \to P_2$ be a linear transformation for which we know that $L(t + 1) = t^2 + t - 2$ and $L(t - 1) = t^2 - t - 1$. Find $L(6t - 4)$.

Note that $\{t + 1, t - 1\}$ is a basis for $P_1$, and that

$$6t - 4 = 1(t + 1) + 5(t - 1).$$

It follows that

$$L(6t - 4) = L((t + 1) + 5(t - 1)) = L(t + 1) + 5L(t - 1) = (t^2 + t - 2) + 5(t^2 - t - 1) = 6t^2 - 4t - 7.$$

2. (a) Find the standard matrix representing the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^4$ where

$$L \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a - b \\ a + c \\ b + c \\ b - a \end{bmatrix}.$$  

We apply the map to the standard basis vectors, obtaining

$$L \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, L \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, L \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$  

It follows that the standard matrix representation for $L$ is given by

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$   

(b) Find a set of vectors spanning the kernel of $L$.

We solve the matrix equation $Ax = 0$, which turns into the following augmented matrix and reduces as shown:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
This corresponds to the system \( a - b = 0, \ b + c = 0 \), which has general solution
\[
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix} = \begin{bmatrix} -r \\ -r \\ r \\ -r \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.
\]
This last vector, therefore, spans the kernel of \( L \).

(c) Find a set of vectors spanning the image of \( L \).

Because the linear map is just matrix multiplication, we may use the fact that the columns of \( A \) span its image. Thus a spanning set is given by
\[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.
\]

3. Define a linear transformation \( L : M_{22} \to P_2 \) by
\[
L \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b + d)t^2 + (b - c)t + (c - d).
\]

(a) Find a set of vectors spanning the kernel of \( L \).

If the output of \( L \) is zero, then the coefficients are all zero. This leads to the linear system
\[
a + b + d = 0, \ b - c = 0, \ c - d = 0,
\]
which has general solution
\[
d = r, \ c = r, \ b = r, \ a = -2r.
\]
Thus the kernel is
\[
\left\{ \begin{bmatrix} -2r \\ r \\ r \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.
\]

(b) Find a set of vectors spanning the image of \( L \).

A typical image vector is of the form
\[
(a + b + d)t^2 + (b - c)t + (c - d) = a(t^2) + b(t^2 + t) + c(-t + 1) + d(t^2 - 1),
\]
so that a spanning set for the image is given by
\[
\{ t^2, t^2 + 1, 1 - t, t^2 - 1 \}.\]
4. Verify the matrix version of the rank-nullity theorem for the following matrix:

\[
A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
1 & 2 & 2 & 1 & 2 \\
2 & 4 & 3 & 3 & 3 \\
0 & 0 & 1 & -1 & -1
\end{bmatrix}
\]

We need to find the nullity of \( A \) (i.e., the dimension of the kernel of the corresponding linear map) and the rank of \( A \) (i.e., the dimension of the image of the corresponding linear map) and make sure they add up to five (the number of columns of \( A \), or the dimension of the domain of the corresponding linear map).

Just as in problem #2, the columns of \( A \) form a spanning set for the image. To find dimension, we need to reduce this spanning set to a basis. To this end we row reduce the matrix \( A \) down to:

\[
\begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The first, third, and fifth columns contain the leading terms, so the first, third, and fifth vectors form a basis. In particular, the dimension of the image is three.

For the kernel, we again row reduce \( A \) (but with a column of zeros adjoined to it) just as before. The system we end up with is:

\[
a + 2b + c + 2d + e = 0, \quad c - d + e = 0, \quad e = 0.
\]

Thus the kernel is:

\[
\begin{bmatrix}
-2s - 3r \\
s \\
r \\
0
\end{bmatrix} = \text{span} \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
1 \\
0
\end{bmatrix}.
\]

Normally, we would again have to find a basis inside this spanning set, but, because it’s only two vectors, it’s easy to see that they are linearly independent (they are not multiples of one another). Thus this is a basis. In particular the dimension of the kernel is two. Three plus two is five, as required.

5. Let \( L : V \to \mathbb{R}^5 \) be a linear transformation.

(a) If \( L \) is one-to-one and \( \dim(\text{Im } L) = 3 \), what is \( \dim(V) \)?

Three (one-to-one implies no dimensions die, so ending with three implies we began with three).
(b) If $\dim(V) = 3$ and $\dim(\ker L) = 1$, what is $\dim(\text{Im } L)$?

Two (we begin with three dimensions, and one dies, so we’re left with two).

(c) If $L$ is onto, what can we say about $\dim(V)$?

$\dim(V) \geq 5$ (the number of dimensions that survive is five, so there must have been at least five to start with).

(d) If $L$ is one-to-one, what can we say about $\dim(V)$?

$\dim(V) \leq 5$ (no dimensions die, and yet the image fits inside $\mathbb{R}^5$, so there must have been no more than five to start with).

(e) If $L$ is one-to-one and onto, what can we say about $\dim(V)$?

$\dim(V) = 5$ (from parts (c) and (d)).