Instructions: Be sure to show as much work as possible, and please make a sincere effort to express your answers clearly and neatly. Please write your answers on your own paper, then staple your pages together using this sheet as a cover sheet.

1. [3 pts] Suppose \( \lambda \) is an eigenvalue of the \( n \times n \) matrix \( A \). Show that the set of all eigenvectors associated with \( \lambda \) (along with the zero vector) is a subspace of \( \mathbb{R}^n \). (This is called the eigenspace associated to \( \lambda \).)

2. (a) [3 pts] Find the characteristic equation for the matrix

\[
A = \begin{bmatrix}
3 & 0 & 0 \\
-2 & 3 & -2 \\
2 & 0 & 5
\end{bmatrix}
\]

(b) [3 pts] Find the eigenvalues for this matrix.

(c) [3 pts] For each eigenvalue, find a basis for the corresponding eigenspace by solving the equation \( Ax = \lambda x \).

(d) [3 pts] Explain why your answers above imply that \( A \) is diagonalizable.

(e) [3 pts] Verify directly that \( A \) is diagonalizable by showing that \( P^{-1}AP = D \). (In particular, find \( P \), \( P^{-1} \), and \( D \).)

(f) [3 pts] The Cayley-Hamilton theorem says that every square matrix satisfies its own characteristic equation. Verify this theorem for \( A \). In other words, if the characteristic polynomial is

\[ p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \]

show that

\[ A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI_n = O. \]

Bonus: Suppose \( C = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \) is a \( 2 \times 2 \) symmetric matrix.

(a) [3 pts] Show that the eigenvalues of \( C \) are real by explicitly computing them.

(b) [3 pts] Show that \( C \) is diagonalizable by showing that either the eigenvalues are distinct, or \( C \) is already diagonal.