1. Let \( L : P_2 \to P_2 \) be defined by \( L(at^2 + bt + c) = (2a - c)t^2 + (a + b - c)t + (c) \).

(a) Find the matrix representation for \( L \) with respect to the basis (on both sides)
\( S = \{t + 1, t^2 + t, t^2 + t + 1 \} \).

We first plug our basis vectors into the function, obtaining the following:
\[
L(t + 1) = -t^2 + 1, \\
L(t^2 + t) = 2t^2 + 2t, \\
L(t^2 + t + 1) = t^2 + t + 1.
\]

We now find the \( S \)-coordinate vectors of these outputs. For the first, we solve
\[
-t^2 + 1 = \alpha(t + 1) + \beta(t^2 + t) + \gamma(t^2 + t + 1) = (\beta + \gamma)t^2 + (\alpha + \beta + \gamma)t + (\alpha + \gamma).
\]
The corresponding augmented matrix is
\[
\begin{bmatrix}
0 & 1 & 1 & -1 \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix},
\]
so that
\[
[L(t + 1)]_S = [-t^2 + 1]_S = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.
\]

For the next two outputs, we could set up and solve a system, but notice that both outputs are simple multiples of single basis vectors, so that we can determine their coordinates by inspection to be
\[
[L(t^2 + t)]_S = [2t^2 + 2t]_S = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}
\]
and
\[
[L(t^2 + t + 1)]_S = [t^2 + t + 1]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

Thus the matrix representation is
\[
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

(b) Suppose \( [v]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Find \( L(v) \) in two ways: once by finding \( v \) and plugging directly into \( L \), and once by doing a matrix multiplication to find \( [L(v)]_S \) and then undoing the coordinates.
(1) Using the given coordinates for \(v\), we find that 
\[
v = 1(t + 1) + 2(t^2 + t) + 3(t^2 + t + 1) = 5t^2 + 6t + 4,
\]
so that, using the given formula for \(L\) directly, we have
\[
L(v) = 6t^2 + 7t + 4.
\]

(2) Applying the matrix we found above, we have 
\[
[L(v)]_S = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} = \begin{bmatrix}
1 \\
3 \\
3
\end{bmatrix}.
\]
Using these coordinates to find \(L(v)\), we have
\[
L(v) = 1(t + 1) + 2(t^2 + t) + 3(t^2 + t + 1) = 5t^2 + 6t + 4.
\]

(c) The transition matrix \(P_{S-T}\) from another basis \(T\) to \(S\) is given by
\[
P_{S-T} = \begin{bmatrix}
1 & 0 & -1 \\
1 & -1 & 2 \\
-1 & 1 & -1
\end{bmatrix}.
\]
Find the transition matrix from \(S\) to \(T\) by inverting this matrix.
\[
\begin{bmatrix}
1 & 0 & -1 \\
1 & -1 & 2 \\
-1 & 1 & -1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -1 \\
0 & -1 & 3 \\
0 & 1 & -2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix}.
\]
Thus the transition matrix from \(S\) to \(T\) is
\[
P^{-1} = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix}.
\]

(d) Use the matrices from parts (a) and (c) to find the matrix representation for \(L\) with respect to \(T\).

The matrix representation of \(L\) with respect to \(T\) is
\[
P^{-1}AP = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 \\
1 & -1 & 2 \\
-1 & 1 & -1
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 3 \\
0 & -1 & 4 \\
0 & 1 & 1
\end{bmatrix}.
\]

2. State and verify the rank-nullity theorem for the linear transformation \(L : M_{22} \rightarrow M_{22}\) defined by
\[
L(A) = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} A - A \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.
\]

(Hint: Let \(A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}\) be a random vector in \(M_{22}\); write out \(L(A)\) explicitly.)
The rank-nullity theorem states that
\[ \dim \ker L + \dim \text{range} L = \dim V. \]

For \( A \) as above, we have that
\[
L(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix}.
\]

To find the kernel, we set
\[
L(A) = \begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

and solve, finding that \( c = 0 \) and \( a = d \). Thus the kernel is all matrices of the form
\[
\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

We deduce that the kernel of \( L \) is spanned by the matrices
\[
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.
\]

Because these matrices are linearly independent, they form a basis for the kernel. In particular, the kernel has dimension 2.

For the range, we have that matrices in the range of \( L \) are of the form
\[
\begin{bmatrix} c & d-a \\ 0 & -c \end{bmatrix} = a \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

It follows that these three matrices span the range. As the first is just the negative of the third, we find that a basis for the range is formed by
\[
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.
\]

In particular, the range is 2-dimensional. These dimensions sum to four, which is the dimension of \( M_{22} \), verifying the rank-nullity theorem.

3. For each of the following eight statements, indicate clearly whether it is true or false. (Assume \( V \) is finite dimensional.) For TWO of the statements, also do the following (in the space at the bottom of the page): If the statement is true, then explain why. If the statement is false, then provide a counterexample.

(a) The set of all continuous functions \( f \) so that \( f''(3) = 0 \) is a subspace of \( C(-\infty, \infty) \).

\[ \text{True. This can be checked directly.} \]

(b) If \( L \) is a linear map, then the image of a linearly independent set of vectors is always linearly independent.
False. The zero map provides a counterexample.

(c) ______ Transition matrices are always invertible.

True. The inverse is the reverse transition.

(d) ______ A matrix is invertible if and only if it row-reduces to the identity matrix.

True. This is how the inverse is found.

(e) ______ If a linear map is 1–1, then its matrix representation cannot contain any zero rows.

False. To be 1–1, each column of the matrix must contain an initial term after row-reducing. A matrix that is taller than it is wide can have an initial term in every column and still have some zero rows.

(f) ______ If \( \dim V = n \), then any set with fewer than \( n \) vectors must be linearly independent.

False. The set could contain multiples of the same vector, for instance.

(g) ______ If \( V \) is a 4-dimensional vector space, then \( V \) must be isomorphic to \( M_{22} \).

True. Any 4-dimensional vector space is isomorphic to \( \mathbb{R}^4 \). In particular, \( M_{22} \) is isomorphic to \( \mathbb{R}^4 \), and so is isomorphic to any 4-dimensional vector space (if \( V \) is isomorphic to \( W \), and \( W \) is isomorphic to \( U \), then \( V \) is isomorphic to \( U \)).

(h) ______ Any two bases for \( V \) must contain the same number of vectors.

True. This is what makes dimension well-defined.