1. Consider the following:

\[ \int_{-2}^{0} \int_{0}^{1+x/2} f(x, y) \, dy \, dx + \int_{0}^{1} \int_{0}^{1-x^2} f(x, y) \, dy \, dx. \]

(a) Carefully sketch the region of integration.

There’s a straight line from the point \((-2, 0)\) to the point \((0, 1)\), and then a parabolic arc from \((0, 1)\) to \((1, 0)\).

(b) Express the sum as just one double integral (rather than as a sum of two) by reversing the order of integration.

\[ \int_{0}^{1} \int_{\sqrt{1-y}}^{\sqrt{1-y^2}} f(x, y) \, dx \, dy \]

2. Express the following integral in spherical coordinates. It may help to sketch the solid region \(E\) and/or the domain \(D\). Do not evaluate the integral.

\[ \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} z \, dz \, dx \, dy. \]

Using \(\sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2}\), we see that the region lies between the standard sphere of radius \(\sqrt{2}\) and the standard right circular cone. In the \(xy\)-plane, the fact that \(-\sqrt{1-y^2} \leq x \leq 0\) tells us that the domain \(D\) is a portion of the left half of the region inside the unit circle. Finally, because \(-1 \leq y \leq 1\), we do in fact have the entire left side of the circle as our \(D\).

This region is a spherical wedge. The fact that we are inside the sphere of radius \(\sqrt{2}\) means that the bounds on \(\rho\) are \(0 \leq \rho \leq \sqrt{2}\). From the picture of \(D\), we know that \(\theta\) has \(\pi/2 \leq \theta \leq 3\pi/2\). Finally, the right circular cone has declination \(\pi/4\), so we know that \(0 \leq \phi \leq \pi/4\). Using the fact that \(z = \rho \cos \phi\), we have

\[ \int_{0}^{\pi/4} \int_{\pi/2}^{3\pi/2} \int_{0}^{\sqrt{2}} (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \int_{0}^{\pi/4} \int_{\pi/2}^{3\pi/2} \int_{0}^{\sqrt{2}} \rho^3 \sin \phi \cos \phi \, d\rho \, d\theta \, d\phi. \]

3. A surface occupies the region \(D\) in the \(xy\)-plane bounded by the curves \(y = 2x^2\) and \(y = x^3\). The temperature of the surface at the point \((x, y)\) is described by \(T(x, y) = x + 2\). Find the average temperature of the surface.
The region is both $x$- and $y$-simple. We will use the fact that it is $y$-simple, so we have that $x^3 \leq y \leq 2x^2$, for $0 \leq x \leq 2$. Thus we have

$$\int_0^2 \int_{x^3}^{2x^2} (x + 2) dy \, dx = \int_0^2 (4x^2 - x^4) \, dx = \frac{32}{3} - \frac{32}{5}.$$  

For the volume, we have

$$\int_0^2 \int_{x^3}^{2x^2} dy \, dx = \frac{16}{3} - \frac{16}{4}.$$  

Thus the average temperature is

$$f_{avg} = \frac{\frac{32}{3} - \frac{32}{5}}{\frac{16}{3} - \frac{16}{4}} = \frac{16}{5}.$$  

4. The region $E$ is bounded by the parabolic cylinder $y = x^2$, the plane $z = y$, the plane $y = 4$, and the $xy$-plane. Express the integral $\iiint_E f(x, y, z) \, dV$ in two different ways, using $dV = dz \, dy \, dx$ and $dV = dx \, dy \, dz$.

As a $z$-simple region, lines in $E$ parallel to the $z$-axis start at the $xy$-plane $z = 0$ and end at the plane $z = y$. We thus have $0 \leq z \leq y$. The shadow of $E$ in the $xy$-plane is exactly the region above the parabola $y = x^2$ below $y = 4$. We thus have $x^2 \leq y \leq 4$ and $-2 \leq x \leq 2$. This gives

$$\iiint_E f(x, y, z) \, dV = \int_{-2}^2 \int_{x^2}^{4} \int_0^y f(x, y, z) \, dz \, dy \, dx.$$  

As an $x$-simple region, lines in $E$ parallel to the $x$-axis start on the back side of the parabolic cylinder $y = x^2$ and end on the front side. We thus have $-\sqrt{y} \leq x \leq \sqrt{y}$. The shadow of $E$ on the $yz$-plane is the triangle formed by the $y$-axis, the line $z = y$, and the line $y = 4$. We thus have $0 \leq y \leq z$ and $0 \leq z \leq 2$. This gives

$$\iiint_E f(x, y, z) \, dV = \int_0^4 \int_z^{\sqrt{y}} \int_{-\sqrt{y}}^y f(x, y, z) \, dx \, dy \, dz.$$  

5. A solid $E$ lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 1$, and below the paraboloid $z = 1 + x^2 + y^2$. Evaluate the integral

$$\iiint_E (x^2 + y^2)^{3/2} \, dV.$$  

The region is $z$-simple, with $1 \leq z \leq 1 + x^2 + y^2$. The domain over which the object sits is the circle where the cylinder hits the $xy$-plane, so it’s $x^2 + y^2 = 1$. This is most easily described in polar coordinates with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus we use cylindrical coordinates to calculate

$$\iiint_E (x^2 + y^2)^{3/2} \, dV = \int_0^{2\pi} \int_0^1 \int_{1+r^2}^{r^3} \, r^3 \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r^6 \, dr \, d\theta = \frac{1}{7} \int_0^{2\pi} \, d\theta = \frac{2\pi}{7}.$$