1. (a) Verify that the $2 \times 1$ matrices of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$ form a subspace of $\mathbb{R}^2$.

Let $u = \begin{bmatrix} a \\ -a \end{bmatrix}$ and $v = \begin{bmatrix} b \\ -b \end{bmatrix}$. Then

$$u \oplus v = \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} b \\ -b \end{bmatrix} = \begin{bmatrix} a + b \\ -a - b \end{bmatrix} = \begin{bmatrix} (a + b) \\ -(a + b) \end{bmatrix}.$$ Matrices of this type are therefore closed under addition. Also for any real number $c$ we have

$$c \odot u = c \begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} ca \\ -ca \end{bmatrix},$$

so matrices of this type are closed under scalar multiplication. It follows that these matrices form a subspace of $\mathbb{R}^2$.

(b) Are the matrices of the form $\begin{bmatrix} x \\ x^2 \end{bmatrix}$ a subspace of $\mathbb{R}^2$? Justify your answer.

No. For example, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, which is not the right form. Thus matrices of this form are not closed under addition, and so they cannot form a subspace.

2. Suppose $V$ is the set of all $2 \times 1$ matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x > 0$. It is easy to show that $V$ is not a vector space using the usual operations of $\mathbb{R}^2$, but it is a vector space with the following operations:

if $u = \begin{bmatrix} a \\ b \end{bmatrix}$ ($a > 0$), $v = \begin{bmatrix} s \\ t \end{bmatrix}$ ($s > 0$), and $c$ is a real number,

then we define

$$u \oplus v = \begin{bmatrix} as \\ b + t \end{bmatrix},$$

and $c \odot u = \begin{bmatrix} ac \\ cb \end{bmatrix}$.

(a) With these operations, what is $0$ in $V$? Explain. (Recall that $0$ is defined to be the vector in $V$ with the property that $u \oplus 0 = u$ for all vectors $u$ in $V$.)

The zero vector is some vector in $W$, say $0 = \begin{bmatrix} x \\ y \end{bmatrix}$ for some $x$ and $y$. We need to find what $x$ and $y$ are. By definition of $0$, we have that $u \oplus 0 = u$ for any $u$, so we have

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ b + y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$
By equating entries, we see that \( x = 1 \) and \( y = 0 \), so that \( \mathbf{0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

(b) Property (5) of vector spaces states that for all real numbers \( c \) and all vectors \( \mathbf{u} \) and \( \mathbf{v} \), we have \( c \odot (\mathbf{u} \oplus \mathbf{v}) = (c \odot \mathbf{u}) \oplus (c \odot \mathbf{v}) \). Verify that this property holds for \( V \) with the operations defined above.

For the left hand side we have
\[
(\mathbf{u} \oplus \mathbf{v}) = \begin{bmatrix} a s \\ b + t \end{bmatrix}
\]
For the right hand side we have
\[
(c \odot \mathbf{u}) \oplus (c \odot \mathbf{v}) = \begin{bmatrix} a^c \\ c(b + t) \end{bmatrix}.
\]
The left and right sides are equal, so the property is verified.

3. Showing your work, find a basis \( S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) for \( \mathbb{R}^3 \) that includes the vectors
\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}
\]

We adjoin the standard basis to the given vectors to form the augmented matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
-3 & 1 & 0 & 1 & 0 \\
6 & -2 & 0 & 0 & 1
\end{bmatrix}.
\]
We then row reduce this matrix obtaining, for instance,
\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 4 & 3 & 1 & 0 \\
0 & 0 & 0 & 2 & 1
\end{bmatrix}.
\]
From this it is clear that the reduced row echelon form will have initial ones in columns one, two and four. Thus vectors one, two and four form a basis for \( \mathbb{R}^3 \). The basis is thus
\[
\begin{bmatrix} 1 \\ -3 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.
\]

4. Let \( W \) be the subspace of \( M_{22} \) consisting of all symmetric \( 2 \times 2 \) matrices.
(a) Find a finite set of matrices in \( W \) which span \( W \). (Note that a symmetric \( 2 \times 2 \) matrix must be of the form \( \begin{bmatrix} x & y \\ y & z \end{bmatrix} \) for some real numbers \( x, y, z \).)
Every matrix in $W$ can be written in the form
\[
\begin{bmatrix} x & y \\ y & z \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

It follows that the set of matrices
\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
spans $W$.

(b) Verify that the following set of vectors in $W$ is linearly independent:
\[
v_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

They are linearly independent precisely if the equation
\[a v_1 + b v_2 + c v_3 = 0\]
has only the trivial solution $a = b = c = 0$. This vector equation is equivalent to the equation
\[a \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\]
which in turn is equivalent to the system of equations
\[
\begin{align*}
a &+ c = 0 \\
a + b & = 0 \\
a + b & = 0 \\
c & = 0
\end{align*}
\]
The fourth equation says that $c = 0$, from which equation one says $a = 0$, from which equations two and three say $b = 0$. We deduce that the vectors are linearly independent.

(c) What dimension is $W$? Explain (based on parts (a) and (b) above).

From part (a), there is a set of three vectors which spans $W$. From this we deduce that the dimension of $W$ is at most 3. From part (b), there is a set of three vectors in $W$ which is linearly independent, from which we deduce that the dimension of $W$ is at least three. Thus the dimension of $W$ is three.

5. For each of the following eight statements, indicate clearly whether it is true or false. For all problems, assume that $V$ is a finite-dimensional vector space.

(a) ______ For any vector $u$ in $V$, it is always true that $0 \odot u = 0$. 

(b) True. A set $S = \{v_1, \ldots, v_k\}$ of linearly independent vectors in $V$ is necessarily a basis for the subspace they span.

True. Let $W$ be the subspace they span. Then the vectors span $W$, and they are linearly independent. Thus they are a basis for the subspace $W$.

(c) False. In particular, if $S_2$ has more vectors than the dimension of $V$, then it surely cannot be linearly independent.

(d) False. They could all be multiples of a single vector, for instance (where $n > 1$).

(e) True. This is a theorem, and it’s the reason that it makes sense to talk about the dimension of $V$.

(f) True. This is a theorem, and we know how to find the basis in the case that $V$ is $\mathbb{R}^n$.

(g) False. For example, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^2$ (think of length one vectors pointing in the positive direction along the $x$ and $y$ axes). The vectors of the form $\begin{bmatrix} x \\ x \end{bmatrix}$ (i.e., the line $y = x$) is a subspace of $\mathbb{R}^2$ that doesn’t even contain either of the basis vectors, so they certainly cannot form a basis for the subspace.

(h) True. The dimension of the null-space of a matrix $A$ (i.e., the kernel of the
associated matrix transformation) is the number of arbitrary constants in the solution to the linear system $Ax = 0$.

True. We said in class that the spanning set obtained by pulling out the constants in the general solution will always form a basis in this type of example. Thus the dimension of the kernel is the number of vectors in a basis, which is the number of vectors in this spanning set, which is the number of arbitrary constants in the general solution.