On all problems, “find the Taylor series” means either (i) find a summation notation formula, or (ii) find the first five non-zero terms explicitly.

1. For this problem you will use the fact that the Taylor series for $e^x$, centered at zero, is

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \]

(a) Find the Taylor series for $e^{-x}$ centered at zero.

We substitute $-x$ in for $x$ in the original power series, to obtain

\[ e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots. \]

(b) The hyperbolic cosine function, or $\cosh(x)$, is defined to be

\[ \cosh(x) = \frac{1}{2} \left( e^x + e^{-x} \right). \]

Use your answer to part (a) along with the given information to find the Taylor series for $\cosh(x)$ centered at zero.

Adding the series for $e^x$ to that for $e^{-x}$, we see that the terms containing odd powers of $x$ cancel, while the terms containing even powers of $x$ double. Dividing by two, we are left with the even powers of $x$. Thus we have that

\[ \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots. \]

2. Find the interval of convergence for the following series (don’t forget the endpoints):

\[ \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = 1 - \frac{3x}{\sqrt{2}} + \frac{9x^2}{\sqrt{3}} - \frac{27x^3}{\sqrt{4}} + \frac{81x^4}{\sqrt{5}} - \cdots. \]

Applying the ratio test, we have

\[ \lim_{n \to \infty} \left| \frac{(-3)^{n+1} x^{n+1} \sqrt{n+2}}{\sqrt{n+1} (-3)^n x^n} \right| = \lim_{n \to \infty} \left| 3x \frac{\sqrt{n+2}}{\sqrt{n+1}} \right| = |3x|. \]

It follows that the series converges for $x$ between $-1/3$ and $1/3$. We now check the endpoints. For $x = 1/3$, we obtain the series

\[ \sum_{n=0}^{\infty} \frac{(-3)^n (1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}. \]
which converges by the alternating series test. For \( x = -1/3 \), we obtain

\[
\sum_{n=0}^{\infty} \frac{(-3)^n(-1/3)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}},
\]

which diverges by limit comparison with a \( p \)-series (for \( p = 1/2 \)). Thus the final interval of convergence is

\[
\left( -\frac{1}{3}, \frac{1}{3} \right).
\]

3. Sketch the traces of the surface described by the equation below. Each trace graph should have at least three different curves on it.

\[ x^2 + y - z^2 = 0 \]

The \( x \)-traces are of the form \( y = z^2 - k^2 \), so these are parabolas opening upward, but shifting down as \( k \) moves away from zero. The \( y \) traces are of the form \( x^2 - z^2 = -k \), so these are standard hyperbolas. The \( z \)-traces are of the form \( y = -x^2 + k^2 \), so these are parabolas opening upward, but shifting down as \( k \) moves away from zero.

Matching:

G: \( x^2 + 4y^2 + 9z^2 = 1 \)
L: \( y^2 = x^2 + z^2 - 1 \)
J: \( y = 2x^2 + z^2 \)
A: \( x^2 + 2z^2 = 1 \)
H: \( y^2 = x^2 + z^2 + 1 \)
M: \( y^2 = x^2 + 2z^2 \)
I: \( y = x^2 - z^2 \)
C: \( r = 2 \)
O: \( \rho = 2 \)
B: \( \theta = \pi/3 \)
F: \( \phi = \pi/3 \)

4. Consider the vectors \( \mathbf{a} = \langle 2, 1, 0 \rangle \) and \( \mathbf{b} = \langle 1, -3, 1 \rangle \). Find the following:

(a) \( \mathbf{a} \cdot \mathbf{b} \);

\[ \mathbf{a} \cdot \mathbf{b} = (2)(1) + (1)(-3) + (0)(1) = -1 \]

(b) \( \mathbf{a} \times \mathbf{b} \);

\[ \mathbf{a} \times \mathbf{b} = \langle 1, -2, -7 \rangle \]
(c) the area of the triangle with \( \mathbf{a} \) and \( \mathbf{b} \) as two of its sides

\[
\text{Area} = \frac{1}{2} \| \mathbf{a} \times \mathbf{b} \| = \frac{3\sqrt{6}}{2}
\]

(d) an equation for the plane through \((1, 2, 3)\) and parallel to both \( \mathbf{a} \) and \( \mathbf{b} \);

Parallel to both \( \mathbf{a} \) and \( \mathbf{b} \) means normal to \( \mathbf{a} \times \mathbf{b} \). Using this normal vector found above and plugging the point in, we obtain

\[
1(x - 1) - 2(y - 2) - 7(z - 3) = 0 \Rightarrow x - 2y - 7z = -24.
\]

(e) parametric equations for the line of intersection between the plane through \((1, 2, 3)\) with normal vector \( \mathbf{a} \) and the plane through \((1, 2, 3)\) with normal vector \( \mathbf{b} \);

The fact that the line is contained in both planes means that it’s perpendicular to both normals, and is thus parallel to \( \mathbf{a} \times \mathbf{b} \), found above. Since both planes go through \((1, 2, 3)\), the line must also go through \((1, 2, 3)\). Plugging in, we obtain

\[
\mathbf{r}(t) = (1 + t, 2 - 2t, 3 - 7t).
\]