Coordinates, Transition Matrices, and Matrix Representations

**Notation:** To simplify things, we’ll assume \( V \) is 3-dimensional and \( W \) is 2-dimensional, just so we can be more explicit in the notation. Of course, all of this generalizes in obvious ways to spaces of any finite dimension.

We let \( S = \{v_1, v_2, v_3\} \) and \( S' = \{v'_1, v'_2, v'_3\} \) be bases for \( V \), and \( T = \{w_1, w_2\} \) and \( T' = \{w'_1, w'_2\} \) be bases for \( W \).

**Remark:** On this handout, the notation \([v]\) (i.e., looks like coordinates but without any subscript specifying a basis) means the coordinates with respect to the standard basis of \( V \) (or \( W \), as the case may be). In particular, if \( V = \mathbb{R}^n \), then \([v]\) means just the original column vector \( v \) itself.

◦ **Coordinate Vector:** \([v]_S = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\)

  MEANS: \( v = a v_1 + b v_2 + c v_3 \)

◦ **Transition Matrix:** \( P_{S-S'}\)

  IS: \( \begin{pmatrix} \vdots \\ [v_1]_S \\ \vdots \\ [v_2]_S \\ [v_3]_S \end{pmatrix} \)

  DOES: \( [v]_S = P_{S-S'} [v]_{S'} \)

  PICTURE:

  \begin{align*}
  \mathbb{R}^3 & \xrightarrow{P} \mathbb{R}^3 & [v]_{S'} & \xrightarrow{P} [v]_S = P [v]_{S'} \\
  \mathbb{R}^3 & \xrightarrow{S'} V & S & \xrightarrow{S} \mathbb{R}^3 \\
  \end{align*}

  IS FOUND BY:

  \begin{pmatrix} \vdots \\ [v_1] \\ [v_2] \\ [v_3] \\ \vdots \\ [v'_1] \\ [v'_2] \\ [v'_3] \end{pmatrix} \sim \begin{pmatrix} I_3 \\ \vdots \\ \vdots \\ \vdots \\ P_{S-S'} \end{pmatrix}
Matrix Representation: A representing \( L : V \to W \) w.r.t. \( S \) and \( T \)

\[
\begin{bmatrix}
[L(v_1)]_T & [L(v_2)]_T & [L(v_3)]_T
\end{bmatrix}
\]

Does: \( A[v]_S = [L(v)]_T \)

Picture:

\[
\begin{array}{ccc}
V & \xrightarrow{L} & W \\
\downarrow{S} & & \downarrow{T} \\
\mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^2
\end{array}
\quad
\begin{array}{ccc}
V & \xrightarrow{L(v)} & [L(v)]_T = A[v]_S \\
\downarrow{[v]_S} & & \downarrow{[L(v)]_T} \\
\mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^2
\end{array}
\]

Is found by:

\[
\begin{bmatrix}
[w_1] & [w_2] & [L(v_1)] & [L(v_2)] & [L(v_3)]
\end{bmatrix} \sim \begin{bmatrix} I_2 & A \end{bmatrix}
\]

Relating Matrix Representations: \( B = Q^{-1}AP \)

\[
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{B} & \mathbb{R}^2 \\
\downarrow{S'} & & \downarrow{T'} \\
\mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^2
\end{array}
\quad
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{L} & \mathbb{R}^2 \\
\downarrow{S} & & \downarrow{T} \\
\mathbb{R}^3 & \xrightarrow{A} & \mathbb{R}^2
\end{array}
\]

\[
[v]_{S'} \xrightarrow{B[v]_{S'}} = [L(v)]_{T'} = B[v]_{S'} = Q^{-1}AP[v]_{S'}
\]

\[
[v]_S \xrightarrow{[L(v)]_T} = A[v]_S
\]