Putnam Seminar 2003

More problems from "Mathematical Miniatures".

Catalan's identity

Catalan's identity is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

(Can you prove it?)

1. Prove the equality

$$\frac{1995}{2}-\frac{1994}{3}+\cdots-\frac{2}{1995}+\frac{1}{1996}=\frac{1}{999}+\frac{3}{1000}+\frac{5}{1001}+\cdots+\frac{1995}{1996}.$$

2. (First problem from the 1979 International Mathematics Olympiad) Let m and n be positive integers such that

$$\frac{m}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1318}+\frac{1}{1319}.$$

Prove that m is divisible by 1979.

If you can do problem **2**, then you should be able to use the same method to prove the following generalization:

2′. Let p be a prime greater than 3, $q=\lfloor 2p/3\rfloor$ (i.e., q is the greatest integer less than or equal to $2p/3$), and let m and n be positive integers such that

$$\frac{m}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{q-1}\frac{1}{q}.$$

Prove that m is divisible by p.

Problem **2′** can in turn be used to solve the following problem, which was on the 1996 Putnam exam:

3. If p be a prime greater than 3 and $q=\lfloor 2p/3\rfloor$, prove that

$$\binom{p}{1}+\binom{p}{2}+\cdots+\binom{p}{q}$$

is divisible by p^2.