Diophantine Equations I

Putnam practice

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In his book *Arithmetica* Diophantus discussed the problem of finding all integral (or rational) solutions of an equation in two or more variables. Fourteen centuries later Fermat was reading Diophantus' book and asked the famous question: show that for $n > 2$

$$x^n + y^n = z^n$$

has no solution, where x, y, z are rational numbers. This question was answered only recently by A. Wiles.

There is no general algorithm for dealing with Diophantine equations. However, there are some techniques that work in particular cases.

Factoring

Problem 1 *Find all pairs (x, y) of integers that satisfy the equation*

$$x(y+1)^2 = 243y$$

Congruence argument

Problem 2 *Prove that $x^2 = 3y^2 + 8$ has no solution in integers x, y.*

Problem 3 *(USAMO 1976) Find all solutions of the equation $a^2+b^2+c^2 = a^2b^2$ in natural numbers a, b, c.*

An integral solution of an equation quadratic in one of the variables is possible only if the discriminant is a perfect square.

Problem 4 *Find all solutions of the equation $x^2 - xy + y = 3$ in integers x, y.*

A homogeneous Diophantine equation like $x^2 + y^2 = z^2$ (x, y, z integers) can be phrased in terms of rational points on a curve.

Problem 5 *Find all the rational points (x, y) on the circle $x^2 + y^2 = 1$.*