**EXAM 2** Name ____________________

**Math 2423**

**4-2-10** Row ____________

**Instructions** *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(12 points)*

a) Give a proof that $\frac{d}{dx}e^x = e^x$.

[12]

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\left(\frac{e^{x+h}-e^x}{h}\right) \; 4 \; = \lim_{h\to 0}\left(\frac{e^x\cdot e^h - e^x}{h}\right) \; 2 \; =$$

$$= \lim_{h\to 0} e^x\left(\frac{e^h-1}{h}\right) \; 2 \; = e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) \; 2 \; = e^x\cdot 1 \; 2 \; = e^x$$

b) Give a proof that $\frac{d}{dx}\ln x = \frac{1}{x}$.

[12]

Let $y = \ln x$ 2

Then $x = e^y$ 2, so $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ 2 $= \frac{1}{e^y}$ 2 $= \frac{1}{x}$ 2.

and $\frac{dx}{dy} = e^y$ 2

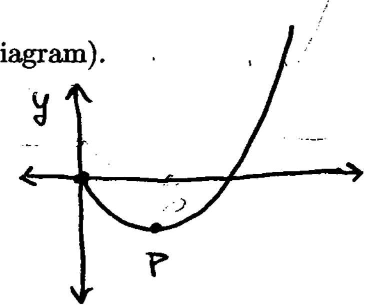
2. *(16 points)*

a) Find the derivative of the function $y = x^{10}\ln x$.

[6]

$$\frac{d}{dx}(x^{10}\ln x) = \underset{2}{10x^9}\cdot\underset{2}{\ln x} + \underset{2}{x^{10}\cdot\frac{1}{x}}$$

b) Find the $x$-coordinate of the lowest point $P$ on the graph of $y = x^{10}\ln x$ (see diagram).




[10] At $P$, $\frac{dy}{dx} = 0$, so

$10x^9\ln x + x^{10}\cdot\frac{1}{x} = 0$, 2

$10x^9\ln x + x^9 = 0$ 2

$x^9(10\ln x + 1) = 0$ 2 $\Rightarrow$ $x^9 = 0$ or $10\ln x + 1 = 0$.

$\Rightarrow x = 0$ or $\ln x = -\frac{1}{10}$ 2 $\Rightarrow x = 0$ or $x = e^{-1/10}$.

$P$ is not at $x = 0$, so $P$ is at $\boxed{x = e^{-1/10}}$ 2

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3. *(24 points)* Find the indefinite integral, showing all work. Remember to express your answer as a function of $x$.

a) $\displaystyle\int \frac{1}{x\,(1+\ln x)}\,dx \;=\; \int \frac{1}{1+\ln x}\cdot\frac{1}{x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$ 2 2 2

[12]

Take
$\boxed{\begin{aligned} u &= 1+\ln x \quad 2 \\ du &= \left(0+\tfrac{1}{x}\right)dx \quad 2\end{aligned}}$

$= \boxed{\ln|1+\ln x| + C}$ 2

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[12] b) $\displaystyle\int (x^2+2x^5)e^{(x^3+x^6)}\,dx \;=\; \int (x^2+2x^5)\,e^{u}\,\frac{du}{(3x^2+6x^5)} = \int \frac{x^2+2x^5}{3(x^2+2x^5)}\,e^{u}\,du$ 2 2

$\boxed{\begin{aligned} u &= x^3+x^6 \quad 2 \\ du &= (3x^2+6x^5)\,dx \quad 2 \\ \frac{du}{3x^2+6x^5} &= dx \end{aligned}}$

$= \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^{u}$ 2 1

$= \boxed{\frac{1}{3}\,e^{(x^3+x^6)} + C}$ 1

5. *(24 points)* Find the value of the definite integral, showing all work. Simplify your answer as much as possible.

[12] a) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^x+1}}\,dx \;=\; \int_2^{e+1} \frac{du}{\sqrt{u}} = \int_2^{e+1} u^{-\frac{1}{2}}\,du = \left[\frac{u^{1/2}}{1/2}\right]_2^{e+1}$ 2 1 1

$\boxed{\begin{aligned} u &= e^x+1 \quad 2 \\ du &= e^x\,dx \\ x=0 &\rightarrow u = e^0+1 = 1+1 = 2 \quad 1 \\ x=1 &\rightarrow u = e^1+1 = e+1 \quad 1 \end{aligned}}$

$= \left[2\sqrt{u}\right]_2^{e+1}$

$= \boxed{2\sqrt{e+1} - 2\sqrt{2}}$ 2

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[12] b) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+1}\,dx \;=\; \int_1^2 \frac{du}{u} = \left[\ln|u|\right]_1^2$ 2 2

$\boxed{\begin{aligned} u &= \sin x+1 \quad 2 \\ du &= (\cos x+0)\,dx = \cos x\,dx \\ x=0 &\rightarrow u = \sin 0+1 = 0+1 = 1 \quad 2 \\ x=\tfrac{\pi}{2} &\rightarrow u = \sin\tfrac{\pi}{2}+1 = 1+1 = 2 \quad 2 \end{aligned}}$

$= \ln 2 - \ln 1$ 1

$= \ln 2 - 0$

$= \boxed{\ln 2}$ 1

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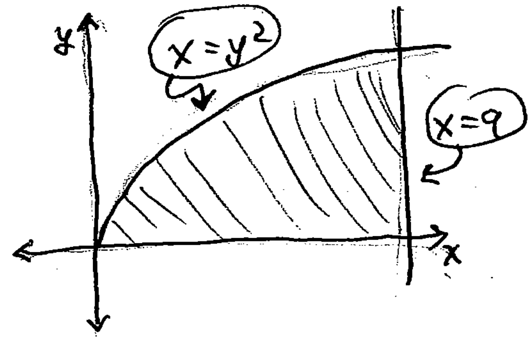
6. *(24 points)* The shaded region in the diagram lies between the curve $x = y^2$, the line $x = 9$, and the line $x = 0$.

[12] a) Find the volume obtained by revolving the region around the $x$-axis.




This can be done by discs or shells.

BY DISCS:

$$V = \int_0^9 \pi y^2\,dx = \int_0^9 \pi x\,dx = \left[\frac{\pi x^2}{2}\right]_0^9$$

$$= \frac{\pi 9^2}{2} = \boxed{\frac{81\pi}{2}}.$$

BY SHELLS:

$$V = \int_0^3 2\pi y(9-y^2)\,dy = \int_0^3 2\pi(9y-y^3)\,dy$$

$$= 2\pi\left[\frac{9y^2}{2} - \frac{y^4}{4}\right]_0^3 = 2\pi\left[\frac{9\cdot 9}{2} - \frac{81}{4}\right] = 2\pi\cdot 81\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= 2\pi\cdot 81\cdot\frac{1}{4} = \boxed{\frac{81\pi}{2}}$$

[12] b) Find the volume obtained by revolving the region around the $y$-axis.

This can also be done by ~~discs~~ washers or by shells.

BY SHELLS:

$$V = \int_0^9 2\pi x\, y\,dx = \int_0^9 2\pi x\sqrt{x}\,dx = 2\pi\int_0^9 x\cdot x^{\frac{1}{2}}\,dx$$

$$= 2\pi\int_0^9 x^{3/2}\,dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^9 = \boxed{\frac{4\pi}{5}\cdot 9^{5/2}}$$

(can be simplified to $\frac{4\pi}{5}(\sqrt{9})^5 = \frac{4\pi}{5}\cdot 3^5 = \frac{4\pi}{5}\cdot 243 = \frac{972\pi}{5}$)

BY WASHERS:

$$V = \int_0^3 \pi\left(9^2-(y^2)^2\right)dy$$

$$= \pi\int_0^3 (81-y^4)\,dy = \pi\left[81y - \frac{y^5}{5}\right]_0^3 = \boxed{\pi\left[243 - \frac{243}{5}\right]}$$

$$= \pi\left[\frac{4\cdot 243}{5}\right] = \frac{972\pi}{5}.$$