Math 4513
Possible presentation topics — part 2

The first list of possible presentation topics which I handed out a couple of weeks ago contained descriptions of some unsolved problems which were mostly taken from plane geometry. Here is another list of unsolved problems, this time mostly taken from number theory. Again, a good reference is *Old and New Unsolved Problems in Plane Geometry and Number Theory* by Victor Klee and Stan Wagon (on reserve in the mathematics library). For a more comprehensive list of unsolved problems in number theory, most of which can be stated in quite elementary terms, see *Unsolved Problems in Number Theory* by Richard K. Guy.

1. It's been known since ancient times that there are lots of integer solutions to the equation $x^2+y^2=z^2$. For example, $3^2+4^2=5^2$ and $5^2+12^2=13^2$; there are infinitely many other such examples. In the 17th century, Pierre de Fermat proved that there are no integer solutions to the equation $x^4+y^4=z^4$, and suggested that indeed there are no integer solutions to the equation $x^n+y^n=z^n$ for any power n which is greater than 2. Fermat's conjecture was finally proved by Andrew Wiles in 1994 (see, e.g., **http://www.pbs.org/wgbh/nova/proof** for the interesting story), but there are many other closely related problems that remain unsolved.

Euler proved in 1753 that it is impossible for the sum of two cubes to equal another cube; that is, there are no natural numbers x, y, and z such that $x^3+y^3=z^3$. (This is a special case of Fermat's conjecture.) Euler then went on to conjecture that it is impossible for the sum of three fourth powers to equal another fourth power, for the sum of four fifth powers to equal another fifth power, and so on. However in recent years, with the aid of computers, this conjecture has been proved wrong for fourth powers and fifth powers: it has been found that

$$95800^4+217519^4+414560^4=422481^4$$

and

$$27^5+84^5+110^5+133^5=144^5.$$

What about sixth powers? Is it possible for the sum of five sixth powers to equal another sixth power? The answer to this question remains unknown. In fact, no one has yet succeeded even in finding *six* sixth powers which add up to another sixth power.

For a list of other unsolved problems relating to sums of powers of numbers, see

http://www.math.niu.edu/ rusin/known-math/index/11PXX.html.

Many of these unsolved problems could be answered if one were able to prove a famous conjecture in number theory called the ABC conjecture. The ABC conjecture is stated easily in elementary terms — you can find a complete explanation in just a few paragraphs at

http://www.maa.org/mathland/mathtrek_12_8.html

— but experts believe a proof of this conjecture lies well beyond the reach of present-day mathematics.

2. If a box has dimensions of x inches by y inches by z inches, then the length of the main diagonal will be $\sqrt{x^2+y^2+z^2}$. The faces of the box will also have diagonals of lengths $\sqrt{x^2+y^2}$, $\sqrt{x^2+z^2}$, and $\sqrt{y^2+z^2}$. Is it possible for all of the numbers x, y, z, $\sqrt{x^2+y^2+z^2}$, $\sqrt{x^2+y^2}$, $\sqrt{x^2+z^2}$, and $\sqrt{y^2+z^2}$ to be integers? In other words, does there exist a box with sides of integer length whose three face diagonals and main diagonal all have integer length as well? Euler (and later others) found examples of boxes with integer side lengths and integer-length face diagonals, but no one has yet found such a box which also has an integer-length main diagonal.

3. A fraction with numerator equal to 1, such as $\frac{1}{5}$ or $\frac{1}{23}$, is called a *unit fraction.* They are also sometimes called Egyptian fractions because the ancient Egyptians — unlike the ancient Greeks — had well-developed methods for computing with fractions that involved reducing all fractions to sums of unit fractions.

Suppose you are given a fraction $\frac{p}{q}$, and, like the ancient Egyptians, want to express it as a sum of unit fractions. You can start by first finding the smallest number n_1 such that $\frac{1}{n_1} < \frac{p}{q}$, and then finding the smallest number n_2 such that $\frac{1}{n_1} + \frac{1}{n_2} < \frac{p}{q}$, and then finding the smallest number n_3 such that $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} < \frac{p}{q}$, and so on. Fibonacci proved in the 1200's that no matter what fraction $\frac{p}{q}$ you start with, this process will always stop after a finite number of steps, at which point you will have

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_k} = \frac{p}{q}$$

for some numbers n_1, n_2, ... , n_k.

What if you restrict all the denominators in the procedure to be odd? In other words, you start with a fraction $\frac{p}{q}$ where q is odd, and then find the smallest odd number n_1 such that $\frac{1}{n_1} < \frac{p}{q}$, and then the smallest odd number n_2 such that $\frac{1}{n_1} + \frac{1}{n_2} < \frac{p}{q}$, and so on. Will this procedure always stop after a finite number of steps? No one knows the answer to this question. (Notice that we are not just asking whether it is possible to express $\frac{p}{q}$ as the sum of unit fractions with odd denominator. Instead, the question is whether *this particular method* for finding such a sum will always work.)

4. Two unsolved problems about perfect numbers, which we mentioned in class, are: does there exist an odd perfect number? and do there exist infinitely many even perfect numbers?

5. Let $\pi(x)$ denote the number of primes less than or equal to x. Thus, for example, $\pi(10) = 4$, because there are 4 primes which are less than or equal to 10 (namely 2, 3, 5, and 7). You can find a graph of $\pi(x)$ in the northwest stairwell of the Physical Sciences Center.

It has been proved that $\pi(x)$ is well approximated by the value of the integral $\int_0^x \frac{1}{\log t}\, dt$, with the approximation getting better and better as x goes to infinity. The famous Riemann hypothesis asks a seemingly quite technical question about how good this approximation is. Namely, can one find a constant C such that for every real number x, the difference between $\pi(x)$ and $\int_0^x \frac{1}{\log t}\, dt$ is less than $C\sqrt{x}\log x$? Despite this question seeming to be rather arbitrary and unmotivated, it turns out that this it has deep ramifications in analysis and number theory. The Riemann hypothesis is probably the most famous unsolved problem in mathematics.

A nice book about the Riemann hypothesis, written at an accessible level, is *Prime Obsession* by John Derbyshire. It's available to read online for free at **http://www.nap.edu/catalog/10532.html**.

6. An algorithm for determining the prime factorization of a number N is said to be a *polynomial-time* algorithm if for every number N, the algorithm accomplishes its task without having to do more than Cd^k additions and multiplications, where d is the number of digits in the number N, and C and k are fixed constants (independent of N). Does such an algorithm exist? (If such an algorithm were to be found, it would defeat the encryption methods currently used to guarantee secure internet communication.)

7. No one has yet been able to determine any pattern in the digits of the decimal expansion of π, or to prove that there is no pattern. The question of what constitutes a pattern is open to interpretation, but one way to interpret the absence of a pattern in the digits would be to say that any combination of digits (such as 6574) occurs no more or less frequently in the decimal expansion of π than any other combination. No result along these lines is currently known. For example, it is not known whether 10% of the digits in the decimal expansion of π are sevens.

8. Is π/e a rational number? It is known that both π and e are irrational, but it is not known whether π is a rational multiple of e. Nor is it known whether $\pi + e$ or $\pi \cdot e$ are rational numbers.

9. Can one find an algorithm which computes the first n digits in the decimal expansion of $\sqrt{2}$ by doing an amount of work which is proportional to n? Thus the amount of work such an algorithm would require to find, say, a million digits would be only ten times the amount of work required to find one hundred thousand digits. You can interpret "amount of work" to mean the total number of additions and multiplications required to find the digits.

Not only is the answer to this question unknown, but it remains unknown if you replace $\sqrt{2}$ by any other irrational square root, or for that matter any irrational root of any polynomial equation with integer coefficients.

10. Euler proved in the 1700's that the infinite series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ adds up to $\frac{\pi^2}{6}$, and the infinite series $1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots$ adds up to $\frac{\pi^4}{90}$. In fact, he found a method for finding similar formulas for any sum of the form $1 + \frac{1}{2^k} + \frac{1}{3^k} + \dots$, provided that k is an even number. No such simple formula has yet been found for the sum of the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \dots$; however it was proved not too long ago that the sum of this series is an irrational number. That raises the question: is the sum of the series $1 + \frac{1}{2^5} + \frac{1}{3^5} + \dots$ an irrational number?