Math 4513
Possible presentation topics

Here is a list, more or less randomly selected, of problems from geometry whose investigation leads to many other interesting topics, not just in mathematics but in other scientific fields as well. (In a couple of days I'll give you a similar list of problems from number theory.)

Some of the problems I just state briefly, others I have written more about because I learned a little about them from presentations given by students in Math 4513 last semester. Most of them are discussed in the book *Old and New Unsolved Problems in Plane Geometry and Number Theory*, listed as [1] in the references below.

1. Draw a closed curve in the plane, which may be smooth (like a circle or ellipse) or which may be a polygon (i.e., has straight edges joined at corners (such as a pentagon). View the inside of the curve as the surface of a billiard table and assume that a billiard ball rolling on the table which hits a wall will bounce off in the usual way (i.e., the angle of incidence equals the angle of reflection). There are a variety of interesting solved and unsolved problems concerning the paths, or trajectories, of billiard balls in such regions [1].

Many unsolved problems deal with *periodic* trajectories of billiard balls. These are trajectories in which the ball returns to the same point from which it started, after bouncing off the edges of the table several times. One can ask whether, given a number k, you can find a periodic trajectory in which the ball bounces off the edge exactly k times. If so, how many of these trajectories are there?

2. Suppose we are given n discs, all with the same radius, at certain locations in the plane (possibly overlapping each other). Is it possible that we can move the discs to new locations, so that the centers of the discs are all closer together than they were before, and yet the total area of the union of the discs is greater than before we moved them together? [1]

It seems that this should be impossible; but it is not so easy to prove. People tried unsuccessfully for a long time, until Bezdek and Connelly finally found a proof in 2001 [2]. The higher-dimensional version of the problem is still unsolved, however. That is, it is not known whether, if we are given n spheres in space, all with the same radius, then pushing the centers of the spheres closer together results in a total volume which is less than or equal to the volume before we pushed them together.

3. Suppose C is a convex set in the plane with the property that every plane set S with diameter equal to 1 can be fit inside C (without having to rotate S). What is the smallest that the area of such a set C can be? [1]

4. If you draw three points in the plane and they are not all on the same line, then they form the vertices of a triangle. However, it's easy to draw four points in the plane so that no three of them are on the same line, and yet they do not form the vertices of a convex quadrilateral. (Try it.) On the other hand, it can be shown that if you draw five points in the plane and no three of them are on the same line, then you can always pick out four of them and connect those four points by line segments to obtain a convex quadrilateral. Thus five points in the plane "guarantee a convex quadrilateral", although four do not.

Let $f(n)$ stand for the greatest number of points in the plane that is not sufficient to "guarantee a convex n-gon". In other words, if we mark $f(n) + 1$ points in the plane, then no matter where these point are, we can always pick out n of them which form the vertices of a convex n-gon. But if we use only $f(n)$ points, then it is possible to locate them so that no n of them determine a convex n-gon. As mentioned above, $f(3) = 2$ and $f(4) = 4$. It is also known that $f(5) = 8$. But the value of $f(n)$ is not known for any $n \geq 6$. The most that is currently known about $f(6)$, for example, is that $16 \leq f(6) \leq 36$. There is a conjecture that in general $f(n) = 2^{n-2}$ (notice that this fits the known values of $f(3)$, $f(4)$, and $f(5)$). Although it has been proved that $f(n) \geq 2^{n-2}$ for all n, we still seem to be pretty far from being able to prove that $f(n) \leq 2^{n-2}$.

The problem of determining $f(n)$ is known as the Erdös-Szekeres problem, or sometimes as the Esther Klein problem or the "Happy End" problem [1,3].

5. Suppose you have r different colors of paint, and you paint each point in the plane with one of these colors. It's not hard to show that if you use only three colors, then no matter how you paint the plane, you will always be able to find at least two points in the plane which are exactly one inch apart and which have the same color. On the other hand, it's also not hard to show that if you have seven colors, then it's possible to paint the plane in such a way that no two points which are exactly one inch apart receive the same color. What if you have only six colors? Would it then be possible to paint the plane, as you did with seven colors, so that no two points which are exactly one inch apart receive the same color? The answer to this question is not known. The answer is also not known if you have five colors, or if you have four colors.

The *chromatic number* of the plane is defined to be the largest number r such that, no matter how you paint the plane using r colors, you can always find two points one inch apart which have the same color. According to what we said above, the chromatic number of the plane is either 3, 4, 5, or 6; but no one yet knows which of these it is. The problem of determining the chromatic number of the plane is sometimes called the Hadwiger-Nelson problem [1,3].

6. Both the Erdös-Szekeres problem (#4) and the Hadwiger-Nelson problem (#5) are related to the following problem, which is not actually a problem in geometry, but belongs instead to the field of mathematics known as combinatorics (among the O.U. faculty, Prof. Özaydin in particular is an expert in this field). Suppose you are inviting a number of guests to a party, some of whom know each other and some of whom do not know each other. You don't know who knows each other and who doesn't, but you want to make sure that among the guests you invite, there will be either (i) a group of m people, each of whom knows all the others in the group, or (ii) a group of n people, none of whom knows any of the others in the group. What is the minimum number of guests you must invite in order to be sure that either (i) or (ii) happens? The number which is the answer to this problem is called $R(m,n)$; the letter R standing for Ramsey, who was the person who first asked the question and proved a famous theorem about it.

What Ramsey proved is that $R(m,n)$ does exist; i.e., for any given value of m and n there is a certain number of guest you can invite to guarantee that one of the two alternatives above will occur. However, the actual value of $R(m,n)$ is so far only known for a few choices of m and n. (See MathWorld or [4].)

The definition of $R(m,n)$ has been stated here in the context of inviting guests to a party to make it easier to explain, but it turns out to be a relevant number in many other contexts.

Here is a problem you might try your hand at (from the 1953 Putnam exam):

The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

Do you see why this problem amounts to proving that $R(3,3) = 6$?

7. Suppose you are given n points in the plane which are not all on the same line. Draw all the lines which connect two (or more) of the points. Will you always be able to find at a set of least $n/3$ of these connecting lines which all intersect in a common point? The answer to this question is not known. The situation is simple if none of the connecting lines contains more than two of the points; in this case, for each point in the set you can find $n-1$ connecting lines which pass through that point, and $n-1 \geq n/3$. The complication of the problem arises due to the fact that some of the connecting lines can contain three or more points.

Problems like the above are called incidence problems; "incidence" being another word for "lying on a line" or "passing through a point". You can find much information about other incidence problems in [1] and chapter 7 of [3]. Here is an interesting incidence problem, posed by Sylvester in 1893: *Show that if you are given n points in the plane which are not all on the same line, then there is at least one line which passes through exactly two of the points (no more and no less).* There is a simple and elegant solution in [1].

8. Is there a polygon which tiles the plane but cannot do so periodically? ("Tiling the plane" means that if you had an infinite collection of tiles in the shape of that polygon, you could fit them together to cover the entire plane without any overlaps or spaces in between. "Tiling the plane periodically" means roughly that the polygon tiles the plane in such a way that if you were to stand above one point of the tiled plane and look down at the tiling, and then move to a certain other point and look down from there, the tiling would look exactly the same to you as it did before you moved.)

The physicist Roger Penrose (famous for writing the popular books "The Emperor's New Mind" and "The Road to Reality" as well as for his contributions to physics) discovered a set of two polygons which tiles the plane but cannot do so periodically. But if there is a single polygon which tiles the plane but cannot do so periodically, it has not yet been found.

There is a vast literature on tiling problems, including a lot of pretty pictures which can be found online. Besides the discussions in [1] and [3], there is a popular account of tiling problems in [5] and a more encyclopedic account in [6].

9. The latest issue of Mathematics Magazine (December 2005) contained two articles explaining how problems that we typically think of as requiring calculus to solve can often be solved by geometry instead. The articles are "The lost calculus: tangency and optimization without limits" by Jeff Suzuki (pp. 339–353) and "Honey, where should we sit?" by John A. Frohliger and Brian Hahn (pp. 379 – 384).

References:

A reasonably good general reference is MathWorld at http://mathworld.wolfram.com. Its coverage of mathematical topics can be shallow and spotty, but this is inevitable given the vastness of the mathematical literature. You might be able to use it to find useful references or other bits of information.

A more comprehensive and professional reference is MathSciNet at http://www.ams.org/mathscinet. This is a subscription service, but it's available from the computers in the computer labs and libraries on the OU campus.

The books and articles below can all be found online or in the O.U. Mathematics Library.

[1] *Old and New Unsolved Problems in Plane Geometry and Number Theory* by Victor Klee and Stan Wagon (on reserve in the Mathematics Library). This contains 24 easily readable discussions of interesting unsolved problems. Each discussion mentions one or two main problems and a number of other related problems, and usually includes simple proofs of partial solutions to the problem.

[2] "Pushing disks apart: the Kneser-Poulsen conjecture in the plane", by K. Bezdek and R. Connelly. Online at http://www.math.cornell.edu/∼connelly/kneser.pdf.

[3] *Research Problems in Discrete Geometry* by P. Brass, W. Moser, and J. Pach, Springer Verlag, 2005. This is a more technical and more comprehensive book than reference [1] above. However, it doesn't require any special background to read, only a certain amount of concentration.

[4] "Small Ramsey Numbers", by S. Radziszowski. Online at http://www.combinatorics.org/Surveys/ds1.pdf.

[5] *Penrose Tiles to Trapdoor Ciphers* by Martin Gardner, W. H. Freeman, New York, 1989.

[6] *Tilings and Patterns* by B. Grünbaum and G. C. Shephard, W. H. Freeman, New York, 1987.