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Penrose Tiling

We first need to consider the definitions of periodic and aperiodic or nonperiodic. A tiling is periodic if the tiling repeats in two independent directions, which can be seen in Figure 1. Aperiodic or nonperiodic tiling does not repeat. This does not imply that the pattern is completely random.

In 1961 Hao Wang conjectured that any set of tiles which can tile the plane can tile it periodically. This conjecture came about from Wang's interest in tiling the plane with unit squares with different colored edges, known as Wang dominoes. This problem was important in symbolic logic. In 1964, this conjecture was proved false by Robert Berger. In Berger's thesis he showed it was possible to construct a set of 20,000 Wang dominoes which tile nonperiodically only. This set was eventually reduced to a set of 92 dominoes. By changing the Wang dominoes into polygonal tiles with projections and slots, you get a set that tile nonperiodically. Raphael Robinson constructed six such "square" tiles that force nonperiodic tiling in Figure 2. It is believed that this is the smallest set of such tiles.

Robert Penrose made a set of tiles, not the square type, that only tile nonperiodically. In 1973 Penrose's original set contained 6 tiles, this can be seen in Figure 3. He was able to reduce the number of tiles to 4 and finally to 2 in 1974. The set of tiles that we will be concentrating on today is Penrose set of 2 tiles known as the kite and dart tiles. This can be seen in Figure 4.

Applications of Penrose tiling can be found in crystallography (the study of how chemical atoms fit together), Quasi-crystals (found in non-scratch coating), art, and product design. Kleenex toilet paper used a Penrose tile pattern to quilt toilet paper because it enabled them to get more on a roll without the paper bunching up. The company was sued and the case was settled out of court.

The kite and dart are created by cutting a rhombus which has internal angles of 72° and 108°. The long diagonal is cut in two according to the golden ratio and then the points are joined to the obtuse angles. The kite and dart are then formed out of two golden triangles. This can be seen in Figure 5.

Figure 5 also shows the ratio of kites to darts and the area of a kite to a dart are both the golden mean, $t = (1 + \sqrt{5})/2$. Since the golden mean and the ratio are irrational, the tiling must be non-periodic since were they rational, the tiling would be periodic and only one representative part of the pattern would be necessary to explain the infinite pattern. However, with non-periodic tiling, there are infinite ways to tile the pieces. The golden mean appears again when one tries to divide the tiling into a unit with an integral number of kites and darts. This is impossible, though, because every tiling has 1.618 times as many kites as darts.

In order to create the non-periodic tiling, the kites and darts may not be placed so as to form a rhombus. Since the golden mean is involved, Penrose tilings show approximate or exact five-fold symmetry. While there are seven ways to form a vertex using kites and darts, only two show exact five-fold symmetry. These are when five kites or five darts surround the vertex. The only way to place the kites and darts so as to create a non-periodic tilings is to align them so that the green and the red arcs drawn onto the darts and kites match the arcs of the tiles which share an edge. When two tiles share an edge in a tiling, the patterns must match at these edges. Figure 6 shows how the tiles match along the edges.

John Horton Conway's "inflation" and "deflation" is used to help prove that Penrose tilings can be used to make an infinite amount of different tilings (Hwang). In order to inflate a tiling, Conway first divided a dart in half along the axis of symmetry. He then placed the shorter sides of the original polygon together creating a different and larger kite and dart Penrose tiling. This can be done an infinite number of times. Each time the previous tiling is inflated, it creates a new tiling, each of which is different from the previous tiling. This can be seen in the Figure 7 (Hwang).If the original Penrose tiling is inflated by n, it is called the n-step inflation. "Inflation can be used to show that the ratio of kites to darts approaches the golden ratio as the plane approaches to infinity," (Hwang). This can be shown in the following equation if the variables are defined as follows: let K = kite, D = dart, K' = one-step inflation of a kite, D' = one-step inflation of a dart. In an inflated kite, there are two kites to two half darts, while in an inflated dart, there is one kite to two half darts. The following equations are derived from the previous information:

$$K' = 2K + D \text{ and } D' = K + D.$$

This can then be written in a linear system as follows:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K \\ D \end{bmatrix} = \begin{bmatrix} K' \\ D' \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = P^2$$

For a two-step inflation following equations are obtained:

$$K'' = 2K' + D' = 5K + 3D$$

$$D'' = K' + D' = 3K + 2D$$

These equations then give the following linear system:

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} K \\ D \end{bmatrix} = \begin{bmatrix} K'' \\ D'' \end{bmatrix}$$

The matrices can then be represented by $M = P^{2n}$ where n is the number of times that the original Penrose tiling has been inflated and where P is the Fibonacci matrix, which is as follows:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$, (Hwang).

The deflation process can be done similarly to the inflation process. In order to deflate the tiling, the kite and dart must be filled with smaller versions of the kite and dart that will exactly fill the original area of the tile. The tiles must be arranged according to the matching rules. By using smaller and smaller tiles, the newly created tiles are not touching the boundary of the tiling. The smaller tiles "can then be surrounded by full-size tiles corresponding to tiles that appear in the scaled-down version," ("Penrose Tiling").By doing this, it creates a new tiling, in which this process can be done over and over and will cover the entire plane ("Penrose Tiling"). This can be seen in four different examples in Figure 8 ("Penrose Tiling").

In order to tile a plane with kites and darts with large pieces, this can be done by starting with the original Penrose tiling, then inflating them an infinite number of times so that any size of plane that you would like to tile can be tiled. It is possible to deflate the tiles so that even the smallest portions of the plane can be tiled. The ratio of kites to darts in a tiling can be represented by the following equation:

$$\frac{F_m}{F_{m-1}}$$

By taking the limit of this equation as m approaches infinity, the golden ratio is obtained. Since the kites and darts are made up of the golden ratio in many different ways, then the ratio of kites to darts is going to be the golden ratio. Since the tilings are based off of the golden ratio, which is irrational, a plane cannot be tiled periodically with irrational tiles.

Works Cited

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Weisstein, Eric W. "Penrose Tiles." From *MathWorld*--A Wolfram Web Resource. <http://mathworld.wolfram.com/PenroseTiles.html >.

Appendix

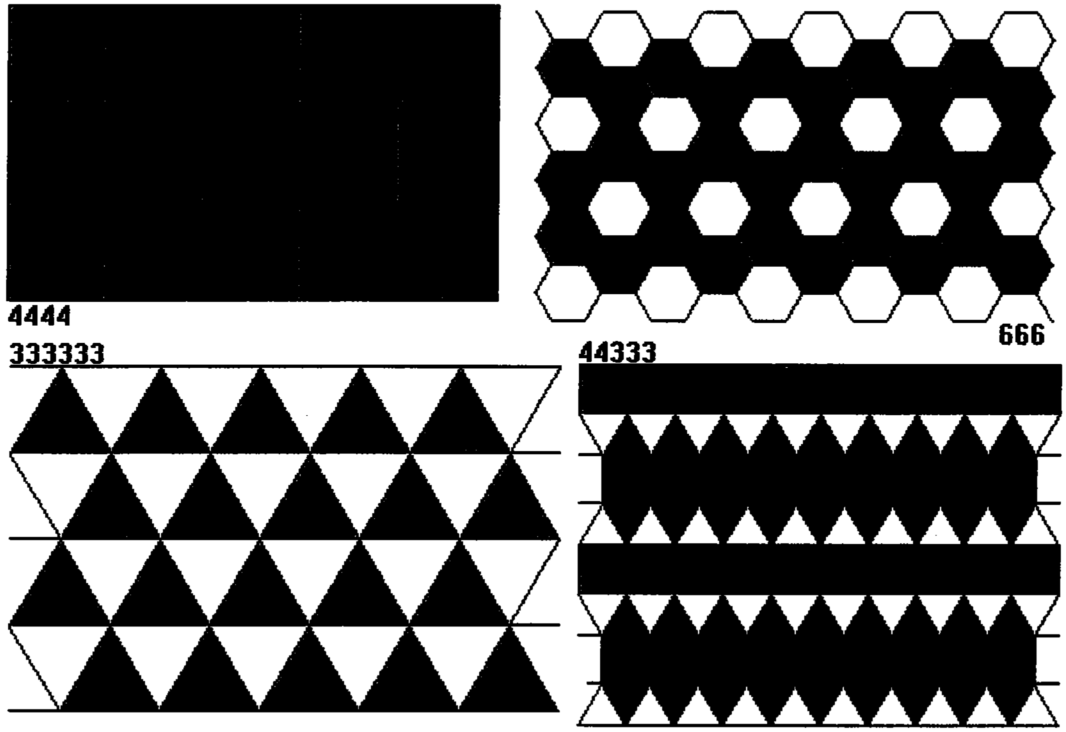


Figure 1. Examples of periodic tiling.

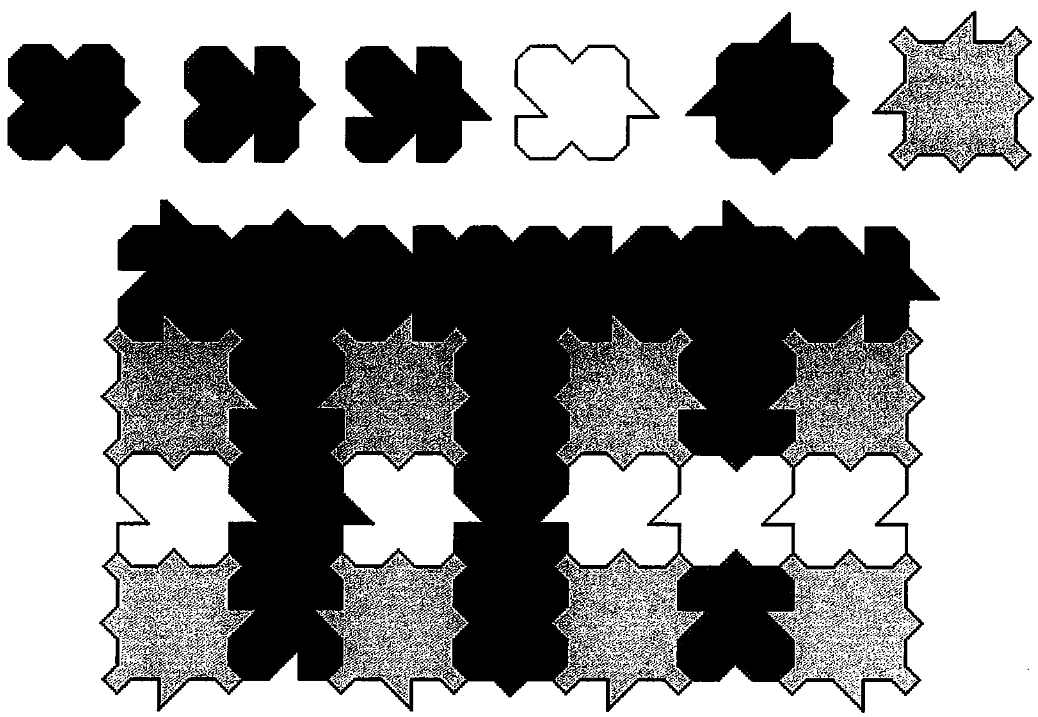


Figure 2. Robinson six “square” tiles. This is also an example of nonperiodic tiling.

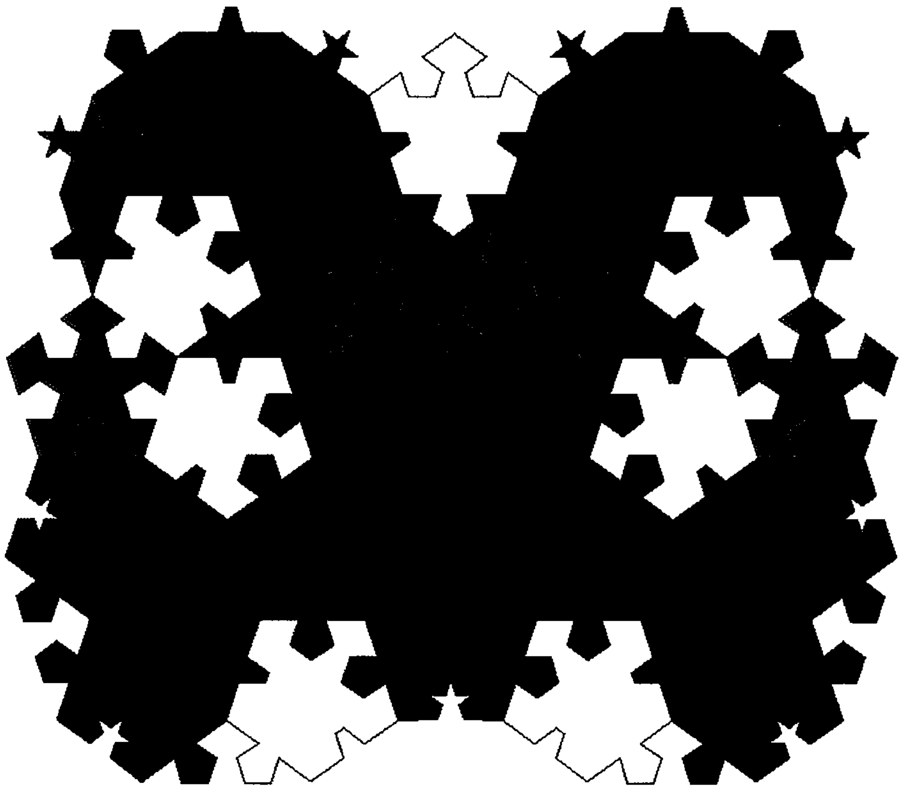


Figure 3. Penrose's six tiles.

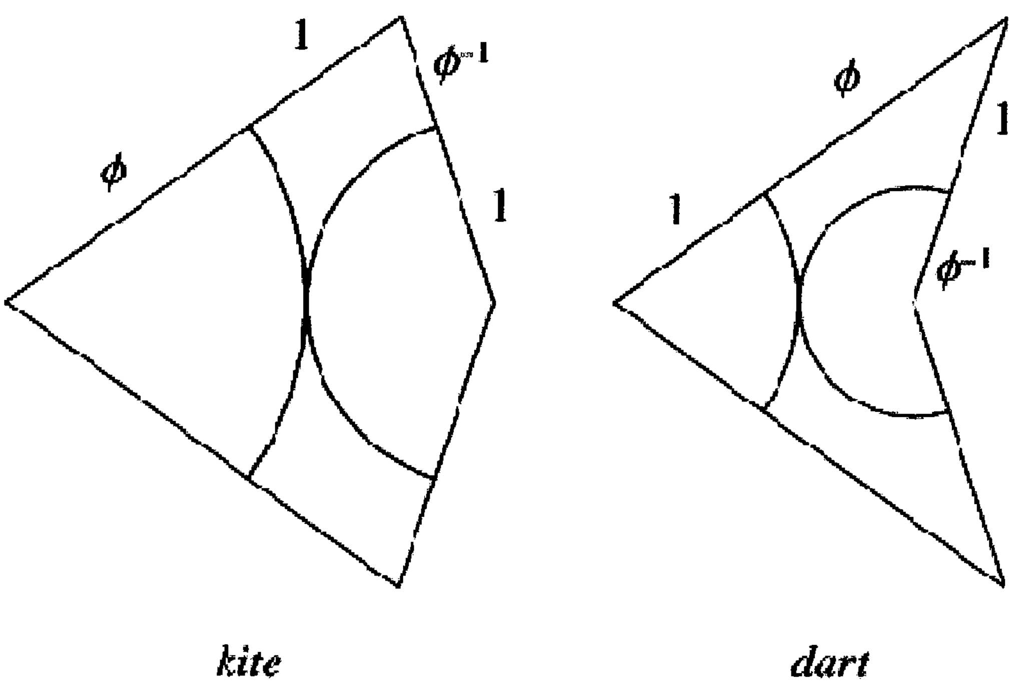
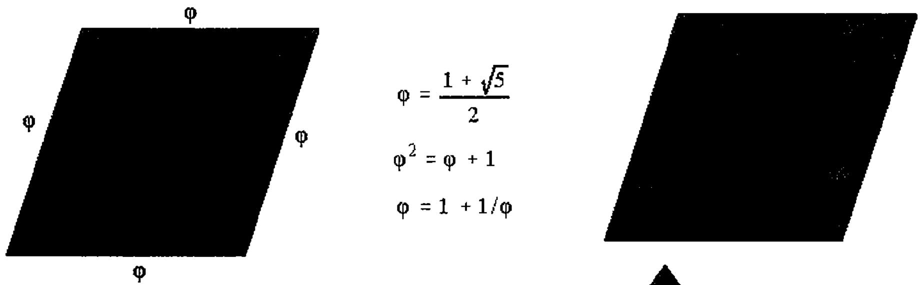


Figure 4. Penrose's kite and dart tiles.



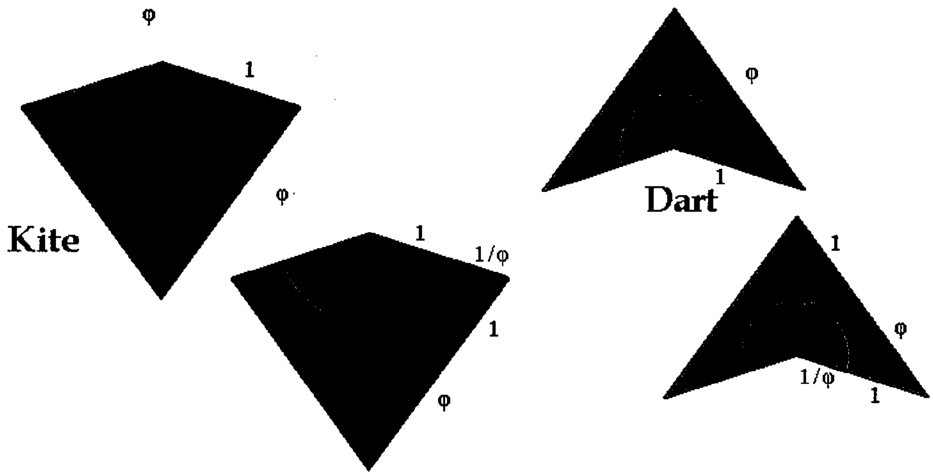


Figure 5. The angles and golden ratio of the kite and dart.

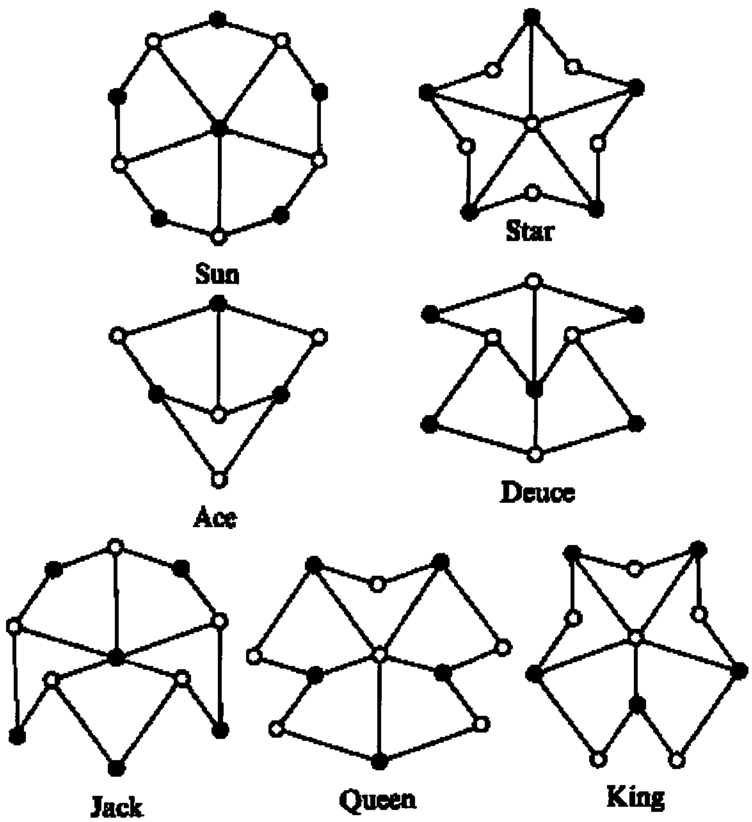


Figure 6. The 7 ways to form a vertex using kites and darts.

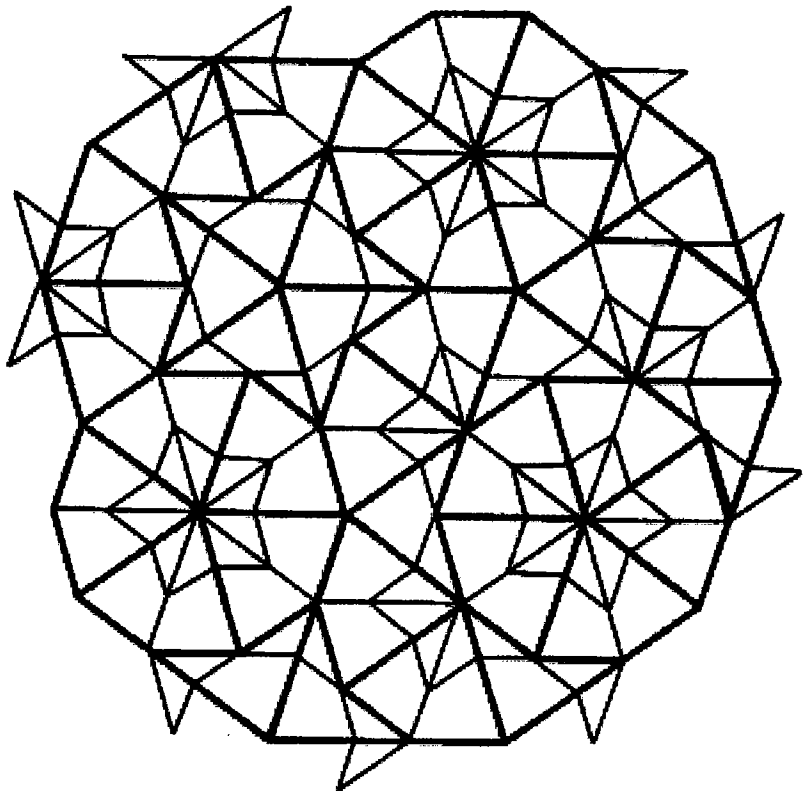


Figure 7. Inflation of Penrose's tiling.

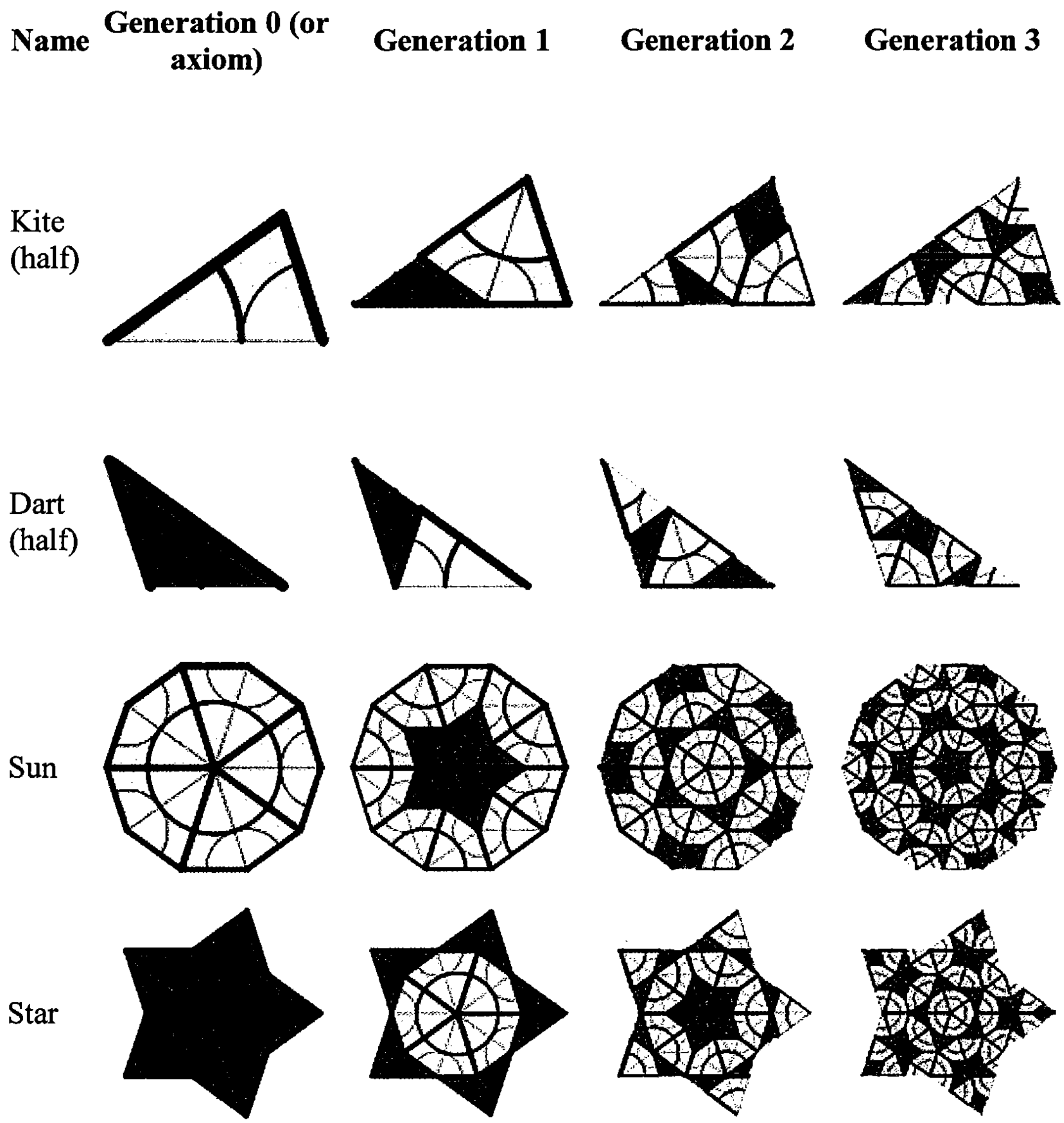


Figure 8. Deflation of four different original Penrose tilings.