**Math 4163**
**Exam 2**

Name: Answer key

**NOTE:** On this exam, you may use any solution formula from class, without having to rederive it.

**1.** *(30 points)* Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions

$$u(0,y) = 0, \qquad u(L,y) = 0, \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,H) = f(x).$$

Write the answer as an infinite series, and express the coefficients in terms of the function $f(x)$.

Write $u(x,y) = P(x)\,Q(y)$ where $\frac{P''(x)}{P(x)} = -\lambda = \frac{-Q''(y)}{Q(y)}$, (4)

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$P(0)=0$, ~~and~~ $P(L)=0$, and $Q(0)=0$. The boundary-value problem

$\begin{cases} P''(x) = -\lambda P(x) \\ P(0)=0 \\ P(L)=0 \end{cases}$ has ~~all~~ eigenvalues $\lambda = +\left(\frac{n\pi}{L}\right)^2$ (2) and eigenfunctions

$P(x) = \sin\left(\frac{n\pi x}{L}\right)$ (2). The equation $Q''(y) = (+\lambda)\,Q(y) = \left(\frac{n\pi}{L}\right)^2 Q(y)$ (2)

has solutions $Q(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$ (2), and from

$Q(0)=0$ we get $A=0$, so $Q(y) = B\sinh\left(\frac{n\pi y}{L}\right)$ (2). Therefore

separated solutions are $u_n(x,y) = B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$ ; $n=1,2,\dots$

Write $\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)}$ (2). Then

$\frac{\partial u}{\partial y}(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cdot\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)$ (2), and

$\frac{\partial u}{\partial y}(x,H) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$ (2) $= f(x)$.

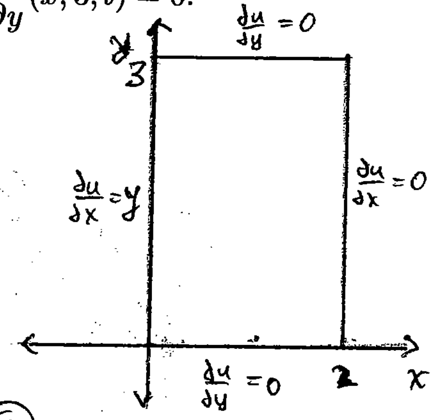
Therefore $B_n\cdot\left(\frac{n\pi}{L}\right)\cdot\cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f(w)\sin\left(\frac{n\pi w}{L}\right)dw$ (2), or

$$\boxed{B_n = \frac{2}{L\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right)}\int_0^L f(w)\sin\left(\frac{n\pi w}{L}\right)dw}$$ (2)

**2.** *(15 points)*

[10] **a.** Suppose $u(x,y,t)$ is a solution of the equation $\frac{\partial u}{\partial t} = \nabla^2 u$ on the rectangle $0 \le x \le 2$, $0 \le y \le 3$. Suppose that at all times $t$, $u$ satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y,t) = y, \qquad \frac{\partial u}{\partial x}(2,y,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,3,t) = 0.$$




(See diagram.) Find the derivative with respect to $t$ of the integral

$$\int_0^3 \int_0^2 u \, dx \, dy.$$

see next page for alternate solution

(Hint: use the divergence theorem.)

$$\frac{d}{dt}\int_0^3\int_0^2 u \, dx\,dy = \int_0^3\int_0^2 \frac{\partial u}{\partial t}\, dx\,dy = \quad (1)$$

$$= \int_0^3\int_0^2 \vec{\nabla}^2 u \, dx\,dy \overset{(1)}{=} \int_0^3\int_0^2 \vec{\nabla}\cdot(\vec{\nabla} u)\,dx\,dy = \oint_C (\vec{\nabla} u\cdot\vec{n}) \quad (2),$$

where the integral $\oint_C$ is over the boundary of the rectangle and $\vec{n}$ is the outward normal to the rectangle. Then

$$\oint_C \vec{\nabla} u\cdot\vec{n} = \int_{\text{left edge}} \vec{\nabla} u\cdot(-\vec{i}) + \int_{\text{top edge}} \vec{\nabla} u\cdot\vec{j} + \int_{\text{right edge}} \vec{\nabla} u\cdot(+\vec{i}) + \int_{\text{bottom edge}} \vec{\nabla} u\cdot(-\vec{j}) = \quad (2)$$

$$= \int_0^3 \left(-\frac{\partial u}{\partial x}(0,y)\right) dy + \int_0^2 \frac{\partial u}{\partial y}(x,3)\,dx + \int_0^3 \frac{\partial u}{\partial x}(2,y)\,dy + \int_0^2 -\frac{\partial u}{\partial y}(x,0)\,dx = \quad (2)$$

$$= \int_0^3 (-y)\,dy + \int_0^2 0\,dx + \int_0^3 0\,dy + \int_0^2 0\,dx = -\frac{9}{2} \quad (2), \text{ So } \boxed{\frac{d}{dt}\int_0^3\int_0^2 u\,dx\,dy = -\frac{9}{2}.}$$

[5] **b.** Does the equation $\nabla^2 u = 0$ have a solution on the rectangle which satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = y, \qquad \frac{\partial u}{\partial x}(2,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,3) = 0?$$

Why or why not?

(2) No, because if $\nabla^2 u = 0$, then $u(x,y)$ is a time-independent solution of $\frac{\partial u}{\partial t} = \nabla^2 u$, with $\frac{\partial u}{\partial t} = 0$. But from part a) it would follow that $\frac{d}{dt}\int_0^3\int_0^2 u\,dx\,dy = \int_0^3\int_0^2 \frac{\partial u}{\partial t}\,dx\,dy = -\frac{9}{2}$ (1), if $u$ satisfied the above boundary conditions. Since $\int_0^3\int_0^2 \frac{\partial u}{\partial t}\,dx\,dy = \int_0^3\int_0^2 0\,dx\,dy = 0$, this is impossible. (2)

Problem 2a, alternate solution:

Use the law of conservation of energy:

$$\frac{d}{dt}\int_0^3\int_0^2 e \, dx\,dy = -\oint_C \vec{\phi}\cdot\vec{n}.$$

Since $e = c\rho u$ and $\vec{\phi} = -K_0 \vec{\nabla} u$, this gives

$$\frac{d}{dt}\int_0^3\int_0^2 c\rho u \, dx\,dy = K_0 \oint_C \vec{\nabla} u\cdot\vec{n},$$

and so $$\frac{d}{dt}\int_0^3\int_0^2 u \, dx\,dy = \frac{K_0}{c\rho}\oint_C \vec{\nabla} u\cdot\vec{n} = k\oint_C \vec{\nabla} u\cdot\vec{n}$$

In this problem, $\frac{\partial u}{\partial t} = k\nabla^2 u = \nabla^2 u$, so $k=1$. Therefore

$$\frac{d}{dt}\int_0^3\int_0^2 u \, dx\,dy = \oint_C \vec{\nabla} u\cdot\vec{n}$$

From here we proceed as in the solution on the preceding page: we use the boundary conditions on $u$ to find that $\oint_C \vec{\nabla} u\cdot\vec{n} = \frac{-9}{2}$,

so $$\frac{d}{dt}\int_0^3\int_0^2 u \, dx\,dy = -\frac{9}{2}.$$