2.5.16 - Solution

(a) Let R stand for the rectangular region in the plane given by $0 \le x \le L$ and $0 \le y \le H$. Let C stand for the four line segments which form the boundary of this region.

According to the law of conservation of heat energy (equation (1.5.1) in the text), if there are no sources ($Q \equiv 0$) then the temperature u within R and the heat flux vector $\vec{\phi}$ on C satisfy

$$(1) \qquad \frac{d}{dt}\iint_R c\rho u \, dV = \oint_C \vec{\phi}\cdot\vec{n}\, d\sigma .$$

Here c is the specific heat, ρ is the density, and $\vec{n}$ is the outward unit vector normal to C. Since this is a two-dimensional problem, the triple integral in (1.5.1) is replaced by a double integral, and the surface integral in (1.5.1) is replaced by a line integral over C, as explained on p. 27 of the text.

In this problem, the temperature is at equilibrium, so $\frac{\partial u}{\partial t} \equiv 0$, and hence $\frac{d}{dt}\iint_R c\rho u \, dV = \iint_R c\rho \frac{\partial u}{\partial t}\, dV = 0.$

Therefore, equation (1) gives

$$(2) \qquad \oint_C \vec{\phi}\cdot\vec{n}\, d\sigma = 0 .$$

Now by Fourier's law of heat conduction, equation (1.5.7) in the text, we have $\vec{\phi} = -K_0 \vec{\nabla} u$, where $-K_0$

is the thermal conductivity. Therefore from (2) we get

$$(3) \qquad \oint_C (\vec{\nabla} u)\cdot\vec{n} = 0 .$$

The line integral in (3) splits into four integrals, one along each of the line segments forming the boundary of R. On the right boundary, $\vec{n} = \vec{i}$, so $\vec{\nabla} u \cdot \vec{n} = \left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j}\right)\cdot\vec{i} =$ $= \frac{\partial u}{\partial x}$. On the left boundary, $\vec{n} = -\vec{i}$, so $\vec{\nabla} u\cdot\vec{n} = -\frac{\partial u}{\partial x}$.

On the top boundary, $\vec{n} = \vec{j}$, so $\vec{\nabla} u\cdot\vec{n} = \frac{\partial u}{\partial y}$. On the bottom boundary, $\vec{n} = -\vec{j}$, so $\vec{\nabla} u\cdot\vec{n} = -\frac{\partial u}{\partial y}$. Therefore, equation (3) becomes:

$$0 = \underbrace{\int_0^L -\frac{\partial u}{\partial y}(x,0)\,dx}_{\text{(bottom)}} + \underbrace{\int_0^H \frac{\partial u}{\partial x}(L,y)\,dy}_{\text{(right)}} + \underbrace{\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx}_{\text{(top)}} + \underbrace{\int_0^H -\frac{\partial u}{\partial x}(0,y)\,dy}_{\text{(left)}}$$

From the given boundary conditions, $\frac{\partial u}{\partial y}(x,0) \equiv 0$, $\frac{\partial u}{\partial x}(L,y) = g(y)$, $\frac{\partial u}{\partial y}(x,H) = f(x)$, and $\frac{\partial u}{\partial x}(0,y) \equiv 0$, so we get

$$\boxed{(4) \quad 0 = \int_0^H g(y)\,dy + \int_0^L f(x)\,dx ,}$$

which is our solvability condition.

Remember that (4) is the same equation as (2), so the physical interpretation of (4) is the same as that of (2): namely that the total amount of heat flowing through the boundary C (per unit time) is 0.