INCOMMENSURABLE LATTICES IN BAUMSLAG–SOLITAR COMPLEXES

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ABSTRACT. This paper concerns locally finite 2–complexes $X_{m,n}$ which are combinatorial models for the Baumslag–Solitar groups $BS(m, n)$. We show that, in many cases, the locally compact group $\operatorname{Aut}(X_{m,n})$ contains incommensurable uniform lattices. The lattices we construct also admit isomorphic Cayley graphs and are finitely presented, torsion-free, and coherent.

1. INTRODUCTION

In this paper we study locally finite CW complexes $X_{m,n}$ which are combinatorial models for the Baumslag–Solitar groups $BS(m, n)$. The group $BS(m, n)$ acts freely and cocompactly on $X_{m,n}$ with quotient space $Z_{m,n}$ homeomorphic to the standard presentation 2–complex of $BS(m, n)$. We wish to understand lattices in the locally compact group $\operatorname{Aut}(X_{m,n})$ of combinatorial automorphisms. $BS(m, n)$ is one such lattice but we are interested in others.

Bass and Kulkarni have shown that if T is a locally finite tree then any two uniform lattices in $\operatorname{Aut}(T)$ are commensurable up to conjugacy [BK90], so in a sense there is only "one" lattice. On the other hand, if T_1 and T_2 are non-isomorphic locally finite trees then $\operatorname{Aut}(T_1 \times T_2) \cong \operatorname{Aut}(T_1) \times \operatorname{Aut}(T_2)$ has a very rich lattice theory, including the celebrated examples of Burger and Mozes [BM00] and Wise [Wis96, Wis07].

The group $\operatorname{Aut}(X_{m,n})$ has some similarities with both $\operatorname{Aut}(T)$ and $\operatorname{Aut}(T_1) \times \operatorname{Aut}(T_2)$, and can be viewed as lying intermediate between the two. Let $T_{m,n}$ denote the regular directed tree in which every vertex has m incoming and n outgoing edges. This is the usual Bass-Serre tree for $BS(m, n)$. The complex $X_{m,n}$ is homeomorphic to $T_{m,n} \times \mathbb{R}$, with projection inducing a homomorphism $\pi_* \colon \operatorname{Aut}(X_{m,n}) \to \operatorname{Aut}(T_{m,n})$. This homomorphism is continuous and *injective* when $m \neq n$ (though not an embedding; see Remark 4.3). Thus, while an element of $\operatorname{Aut}(T_1 \times T_2)$ is an arbitrary pair of tree automorphisms, an element of $\operatorname{Aut}(X_{m,n})$ is just a single one.

Nevertheless, our main result shows that for many values of m and n, $\operatorname{Aut}(X_{m,n})$ is large enough to accommodate incommensurable uniform lattices, just as $\operatorname{Aut}(T_1 \times T_2)$ does. Our main construction appears in Theorem 6.4, restated as follows.

Theorem 1.1. *If* $\gcd(k,n) \neq 1$ *then* $\text{Aut}(X_{k,kn})$ *contains uniform lattices* G_1, G_2 *that are not abstractly commensurable.*

A similar statement holds for other cases of $\text{Aut}(X_{k,kn})$, by a different construction. Combining Theorems 7.1 and 7.2 we have:

Theorem 1.2. *If either*

   (1) *n has a non-trivial divisor $p \neq n$ such that $p < k$, or*
   (2) *$n < k$ and $k \equiv 1 \mod n$*

then $\text{Aut}(X_{k,kn})$ *contains uniform lattices that are not abstractly commensurable.*

The lattices in the previous results are all torsion-free. A combinatorial characterization of the torsion-free lattices in $\text{Aut}(X_{m,n})$ is given in Theorem 4.7.

In the other direction, there is just one situation in which we can say for sure that incommensurable lattices do not exist, namely, when $\text{Aut}(X_{m,n})$ is discrete. This occurs if and only if $\gcd(m,n) = 1$, by Theorem 4.8.

It is perhaps not surprising that the incommensurable lattices of Theorems 1.1 and 1.2 lie in $\text{Aut}(X_{m,n})$ for which m divides n. This behavior illustrates the large degree of symmetry enjoyed by these complexes. By the same token, the groups $BS(k,kn)$ are unusually symmetric. When $k, |n| > 1$, the groups $\text{Aut}(BS(k,kn))$ and $\text{Out}(BS(k,kn))$ fail to be finitely generated, by [CL83] (see also [Cla09] for a newer, more geometric proof).

Methods. Every torsion-free lattice in $\text{Aut}(X_{m,n})$ is a generalized Baumslag–Solitar group (GBS group). These are the fundamental groups of graphs of groups in which all edge and vertex groups are $\mathbb{Z}$. We make use of many of the standard tools for studying these groups, such as deformations and covering theory for finite-index subgroups.

The main new tool developed here is a commensurability invariant called the *depth profile*. It is a subset of $\mathbb{N}$ defined relative to a choice of elliptic subgroup V. It records the manner in which certain conjugates V^g meet V. If two GBS groups are commensurable then their depth profiles are the same, provided both groups contain V. To accommodate changes in V, we impose an equivalence relation on the set of subsets of $\mathbb{N}$. In this way we obtain a well defined invariant of GBS groups; see Section 5. Note that this is an invariant of *abstract* commensurability.

Computing the depth profile of a GBS group is not completely straightforward and we do it only under certain assumptions. It becomes helpful to pass to finite index subgroups in order to meet these assumptions; this is where the covering theory comes in.

One case where the depth profile can be computed easily is the case of Baumslag–Solitar groups. Using the depth profile we obtain a new and much simpler proof of one of the main results of [CRKZ21], in which it is shown that certain Baumslag–Solitar

groups are not commensurable. It is precisely the most difficult case of that result that is handled easily using the depth profile. See Example 5.3 and Corollary 5.4.

Section 6 is the heart of the paper, in which we present the main examples and compute their depth profiles. In Section 7 we present the additional examples comprising Theorem 1.2.

In Section 8 we show that the incommensurable lattices of Theorems 1.1 and 1.2 all admit isomorphic Cayley graphs; see Corollary 8.2. This result is not completely obvious since their actions on $X_{m,n}$ are not transitive on the vertices. However, it is enough that they act on the vertices with the same orbits, which is what happens here.

There are many basic questions remaining regarding lattices in $\text{Aut}(X_{m,n})$, such as determining the pairs m, n for which incommensurable lattices exist. We have said nothing about lattices with torsion, either. Also, there is a phenomenon (see Figure 4 and Remark 6.7) of different groups $\text{Aut}(X_{m,n})$ and $\text{Aut}(X_{m',n'})$ admitting isomorphic lattices in an unexpected way. These questions are gathered together in Section 9.

The motivation for this work came in part from the following question of Stark and Woodhouse [SW19, Question 1.9]: If H and H' are one-ended residually finite hyperbolic groups that act geometrically on the same simplicial complex, are H and H' abstractly commensurable? The groups considered here are certainly not hyperbolic or residually finite, but they provide another setting, in addition to lattices in products of trees, in which incommensurable groups can have a common combinatorial model. See also [DK23] for another recent example of this phenomenon.

2. PRELIMINARIES

2.1. **Graphs.** A *graph* A is a pair of sets $(V(A), E(A))$ with maps $\partial_0, \partial_1 \colon E(A) \to V(A)$ and a free involution $e \mapsto \overline{e}$ on $E(A)$, such that $\partial_i(\overline{e}) = \partial_{1-i}(e)$ for all e. An element of $E(A)$ is an oriented edge with initial vertex $\partial_0(e)$ and terminal vertex $\partial_1(e)$; then $\overline{e}$ is the "same" edge with the opposite orientation. An edge is a *loop* if $\partial_0(e) = \partial_1(e)$. For each $v \in V(A)$ we define $E_0(v) = \{e \in E(A) \mid \partial_0(e) = v\}$.

A *directed graph* is a graph A together with a partition $E(A) = E^+(A) \sqcup E^-(A)$ that separates every pair $\{e, \overline{e}\}$. The edges in $E^+(A)$ are called *directed* edges. The "direction" is from $\partial_0(e)$ to $\partial_1(e)$. For each $v \in V(A)$ we define $E_0^+(v) = E^+(A) \cap E_0(v)$ and $E_0^-(v) = E^-(A) \cap E_0(v)$.

2.2. *G*–**trees.** By a *G*–*tree* we mean a simplicial tree X with an action of G by simplicial automorphisms, without inversions. Given X, if $g \in G$ fixes a vertex we call g *elliptic*. If it is not elliptic, then there is a g–invariant line in X, called the *axis* of g, on which g acts by a non-trivial translation. In this case, we call g *hyperbolic*. An *elliptic subgroup* is a subgroup $H < G$ that fixes a vertex.

2.3. **Automorphisms of CW complexes.** Let X be a CW complex. A *topological automorphism* of X is a homeomorphism which preserves the cell structure, and in particular, the partition of X into open cells. The group of such automorphisms will be denoted by $\text{Aut}_{top}(X)$.

We are interested in a more combinatorial notion, however. The *combinatorial automorphism group*, denoted by $\text{Aut}(X)$, is the quotient of $\text{Aut}_{top}(X)$ in which two automorphisms are considered the same if they induce the same permutation on the set of cells of X.

In many cases one can lift $\text{Aut}(X)$ to a subgroup of $\text{Aut}_{top}(X)$ by imposing a metric constraint on topological automorphisms. For the complexes $X_{m,n}$ considered in this paper, there are piecewise hyperbolic or Euclidean metrics available that can be used in this way; see Section 4.2.

2.4. **Locally finite complexes.** If X is connected and locally finite then $\text{Aut}(X)$ is a locally compact group, as we explain here. For any CW complex X the topology on $\text{Aut}(X)$ has a subbasis given by the sets

$$U^X_{\sigma \to \tau} \ = \ \{ f \in \text{Aut}(X) \mid f(\sigma) = \tau \}$$

for all pairs of cells σ, τ of X. A basis is given by finite intersections of these sets. That is, two automorphisms are close if they agree on a large finite collection of cells of X.

Define two cells to be *adjacent* if their closures intersect nontrivially. For any cell σ let $B_\sigma(r)$ denote the combinatorial ball of radius r about σ. This is defined recursively as follows: $B_\sigma(0) = \{\sigma\}$, and $\tau \in B_\sigma(r)$ if τ is equal or adjacent to a cell in $B_\sigma(r-1)$. In a locally finite CW complex, every cell is adjacent to only finitely many cells, and each $B_\sigma(r)$ is finite. If X is connected then $X = \bigcup_r B_\sigma(r)$ for any σ.

Now let $G = \text{Aut}(X)$ with X connected and locally finite. If σ is a cell, its stabilizer G_σ is open (it equals $U^X_{\sigma \to \sigma}$) and has the structure

$$G_\sigma \ = \ \varprojlim_r \text{Im}\big(G_\sigma \xrightarrow{\text{res}} \text{Aut}(B_\sigma(r))\big).$$

As an inverse limit of finite groups, G_σ is profinite. Since compact open subgroups are always commensurable, all cell stabilizers are commensurable. Finally, the existence of compact open subgroups implies that $\text{Aut}(X)$ is locally compact.

2.5. **Lattices.** In a locally compact group G, a discrete subgroup $\Gamma < G$ is called a *lattice* if G/Γ carries a finite positive G-invariant measure, and a *uniform lattice* if G/Γ is compact.

Now let X be a connected, locally finite CW complex. Let $G = \text{Aut}(X)$. A subgroup $\Gamma < G$ is discrete if and only if every cell stabilizer Γ_σ is finite. In this case, define the *covolume* of Γ to be

$$\text{Vol}(X/\Gamma) = \sum_{[\sigma] \in \text{cells}(X/\Gamma)} 1/|\Gamma_\sigma|.$$

The sum is taken over a set of representatives of the Γ–orbits of cells of X.

The next result follows directly from [BL01, 1.5–1.6].

Proposition 2.1. *Let X be a connected locally finite CW complex. Suppose that $G = \text{Aut}(X)$ acts cocompactly on X and let $\Gamma < G$ be a discrete subgroup. Then*

(1) Γ *is a lattice if and only if* $\text{Vol}(X/\Gamma) < \infty$
(2) Γ *is a uniform lattice if and only if* X/Γ *is compact.*

Note that it follows from 2.1 that every torsion-free lattice is uniform. (Indeed, every lattice with a bound on the size of finite subgroups is uniform.) The lattices considered in this paper will generally be torsion-free.

3. GENERALIZED BAUMSLAG–SOLITAR GROUPS

3.1. **Definitions.** A *generalized Baumslag–Solitar group* (or *GBS group*) is a group that admits a graph of groups decomposition in which all vertex and edge groups are $\mathbb{Z}$. Equivalently, it is a group that acts without inversions on a simplicial tree such that all vertex and edge stabilzers are infinite cyclic. Note that some authors require that GBS groups be finitely generated. We have no such requirement here. We refer to [For06, For03, Lev07] for general background on these groups.

If G is a GBS group with corresponding Bass–Serre tree X, we call X a *GBS tree* for G. Its quotient graph of groups has every edge and vertex group isomorphic to $\mathbb{Z}$, with inclusion maps given by multiplication by various non-zero integers. Thus this graph of groups is specified by a connected graph A ($= X/G$) and a *label function* $\lambda \colon E(A) \to (\mathbb{Z} - \{0\})$. We denote the corresponding graph of groups by $(A, \lambda)_{\mathbb{Z}}$. Letting $\lambda \colon E(X) \to (\mathbb{Z} - \{0\})$ denote the induced label function on X, we have

$$|\lambda(e)| = [G_{\partial_0(e)} : G_e] \tag{3.1}$$

for all edges $e \in E(X)$.

3.2. **Fibered** 2–**complexes.** It is often useful to take a topological viewpoint as in [SW79]. A GBS group is then the fundamental group of the total space of a graph of spaces in which each edge and vertex space is a circle. Given a labeled graph (A, λ) there are oriented circles C_v for each vertex v and $C_e = C_{\overline{e}}$ for each $e \in E(A)$. For each e let M_e be the mapping cylinder of the degree $\lambda(e)$ covering map $C_e \to C_{\partial_0(e)}$. Note that M_e contains embedded copies of both C_e and $C_{\partial_0(e)}$. The total space of the graph of spaces is now defined as the quotient

$$Z_{(A,\lambda)} \;=\; \bigsqcup_{e \in E(A)} M_e \Big/ \sim$$

where all copies of C_v (in M_e for $e \in E_0(v)$) are identified by the identity map, and C_e and $C_{\overline{e}}$ (in M_e and $M_{\overline{e}}$) are identified by the identity, for all $v \in V(A)$ and $e \in E(A)$. We will call $Z_{(A,\lambda)}$ the *fibered* 2–*complex* associated to (A, λ). It is naturally foliated by circles, and may be thought of as a 2–dimensional analogue of a Seifert fibered space, with singular fibers the vertex circles C_v. The leaf space of $Z_{(A,\lambda)}$ is A (realized topologically as a 1–complex). Each M_e is embedded in $Z_{(A,\lambda)}$, and under the map to the leaf space, M_e is the pre-image of a closed half-edge in A.

Example 3.2. Let M be a Seifert fibered 3–manifold that is not closed. Let Σ be the quotient 2–orbifold, which has only isolated cone singularities. The underlying surface (also non-closed) contains a *spine*, i.e., a 1–complex embedded as a deformation retract. This spine S may be chosen so that the singularities of Σ are all vertices of S. The union of the fibers over S is then a deformation retract of M, and it has the structure of a fibered 2–complex. Hence $\pi_1(M)$ is a GBS group.

Other well known examples of GBS groups include free-by-cyclic groups $F_n \rtimes_\varphi \mathbb{Z}$ with periodic monodromy φ [Lev15] and one-relator groups with non-trivial center [Pie74]. Every finite index subgroup of a GBS group is a GBS group.

Example 3.3. Let (A, λ) be the labeled graph having one vertex and one loop e with labels $\lambda(e) = m$, $\lambda(\overline{e}) = n$. The corresponding GBS group is the *Baumslag–Solitar group*

$$BS(m, n) \;=\; \langle\, a, t \mid t a^m t^{-1} = a^n \,\rangle. \tag{3.4}$$

This is the *standard* GBS structure for $BS(m, n)$. In this case we give the space $Z_{(A,\lambda)}$ the additional name $Z_{m,n}$.

Notation 3.5. We denote by $\bigvee_{i=1}^{k} BS(m_i, n_i)$ the GBS group defined by the labeled graph having one vertex and k loops, each with labels m_i and n_i.

3.3. **Non-elementary GBS groups.** A GBS group G is *elementary* if it is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or the Klein bottle group, or is the union of an infinitely ascending chain of infinite cyclic groups $C_0 \subset C_1 \subset C_2 \subset \cdots$. In the last case, G is not finitely generated. Being elementary characterizes the property that some (equivalently, every) GBS tree for G has limit set of cardinality 0, 1, or 2. The three finitely generated cases correspond to limit sets of size 0 or 2 by [For03, Lemma 2.6]. A GBS tree with one-point limit set yields the given description of G as an ascending union by [Bas93, 7.2] or [Tit70, (3.4)].

A fundamental property of GBS groups other than $\mathbb{Z} \times \mathbb{Z}$ and the Klein bottle group (and therefore of all non-elementary GBS groups) is that any two GBS trees for the same group G define the same partition of G into elliptic and hyperbolic elements [For03, Lemmas 2.5, 2.6], and moreover have the same elliptic subgroups.

3.4. **Segments and index.** Let X be a locally finite G–tree. An *edge path* of length k is either a vertex v_0 (if $k = 0$) or a sequence of edges $\sigma = (e_1, \ldots, e_k)$ with $\partial_1(e_i) = v_i = \partial_0(e_{i+1})$ for each i. The *initial* and *terminal vertices* are $\partial_0(\sigma) = v_0$ and $\partial_1(\sigma) = v_k$ respectively. The *reverse* of σ is $\overline{\sigma} = (\overline{e}_k, \ldots, \overline{e}_1)$.

A *segment* is an edge path with no backtracking, meaning that $e_{i+1} \neq \overline{e}_i$ for $1 \leq i < k$. We call a segment *non-trivial* if its length is positive. The pointwise stabilizer of σ is $G_\sigma = G_{\partial_0 \sigma} \cap G_{\partial_1 \sigma}$. If x, y are vertices of X then $[x, y]$ denotes the segment with initial vertex x and terminal vertex y. The *index* of σ is the number $i(\sigma) = [G_{\partial_0 \sigma} : G_\sigma]$. This index is also defined for edges, since an edge is a segment of length one. Note that length zero segments have index 1.

If X is a GBS tree with label function λ induced by the quotient graph of groups, then $i(e) = |\lambda(e)|$ for all $e \in E(X)$, as noted in (3.1). The main difference between *labels* and *indices* is that labels may be negative.

In order to compute indices of segments we will use the following result, which is Corollary 3.6 of [For06]. The original statement had edges and labels instead of segments and indices, but the proof is the same.

Lemma 3.6. *Let X be a GBS tree and suppose the segment σ is a concatenation of non-trivial segments $\sigma = \sigma_1 \cdots \sigma_k$. Let $n_i = i(\sigma_i)$ and $m_i = i(\overline{\sigma}_i)$ for each i. Then G_{σ_1} fixes σ if and only if*

$$\prod_{i=2}^{\ell} n_i \ \text{ divides } \ \prod_{i=1}^{\ell-1} m_i \tag{3.7}$$

for all $\ell = 2, \ldots, k$. $\qquad\qquad\square$

Note: the lemma does not apply to every concatenation of segments. The concatenation itself must also be a segment.

The index of a segment can now be determined as follows (for instance, by taking each σ_i to be an edge):

Lemma 3.8. *Let X be a GBS tree and suppose the segment σ is a concatenation of non-trivial segments $\sigma = \sigma_1 \cdots \sigma_k$. Let $n_i = i(\sigma_i)$ and $m_i = i(\overline{\sigma}_i)$ for each i. Then $i(\sigma)$ is the smallest positive integer r such that*

$$n_1 \mid r \ \ and \ \ \prod_{i=1}^{\ell} n_i \ \ divides \ \ r \prod_{i=1}^{\ell-1} m_i \ \ for \ \ell = 2, \ldots, k. \tag{3.9}$$

Proof. Write $G_{\partial_0 \sigma} = \langle x \rangle$. We want r such that $G_\sigma = \langle x^r \rangle$, which occurs if and only if r is smallest such that $\langle x^r \rangle$ fixes σ. Consider a longer segment $\sigma_0 \cdot \sigma$ where $i(\overline{\sigma}_0) = r$ (perhaps in a larger GBS tree). Then $G_{\sigma_0} = \langle x^r \rangle$, and so $\langle x^r \rangle$ fixes σ if and only if condition (3.7) holds for the concatenation $\sigma_0 \cdots \sigma_k$ (for $\ell = 1, \ldots, k$). These conditions are exactly statement (3.9). $\square$

Remark 3.10. When $k = 2$ there is a closed formula $i(\sigma) = n_1 n_2 / \gcd(m_1, n_2)$. Applying this formula iteratively is usually easier than using (3.9).

More important than (3.9), perhaps, is that $i(\sigma)$ and $i(\overline{\sigma})$ are determined completely by the indices seen along σ, in any expression as a concatenation.

Lemma 3.11. *Let X be a GBS tree and suppose $\sigma = [x, gx]$ for some vertex x with $x \neq gx$. If $i(\sigma) > 1$ then there is a hyperbolic element $h \in G$ such that $hx = gx$.*

Proof. If g is hyperbolic there is nothing to prove. So suppose g is elliptic with fixed subtree X_g. Then $X_g \cap \sigma = \{m\}$, the midpoint of σ. Let $G_x = \langle a \rangle$. If a fixes m then $G_x \subset G_m$, and since $g \in G_m$ we have $G_x = (G_x)^g = G_{gx}$. But then $i(\sigma) = 1$, a contradiction. So $X_g \cap X_a = \emptyset$ and therefore ga is a hyperbolic element taking x to gx. $\square$

3.5. **The modular homomorphism.** Let G be a GBS group with GBS tree X and quotient labeled graph (A, λ). Fix a non-trivial elliptic element $a \in G$. Since elliptic elements are all commensurable, every element g satisfies a relation $g^{-1} a^m g = a^n$ for some non-zero integers m, n. The function $q(g) = m/n$ defines a homomorphism $G \to \mathbb{Q}^\times$ called the *modular homomorphism* (cf. [Kro90, BK90]). It does not depend on the choice of a.

The modular homomorphism takes the value 1 on every elliptic element, so it factors through $\pi_1(A)$, indeed through $H_1(A)$ since $\mathbb{Q}^\times$ is abelian. It can be computed from (A, λ) as follows. If $g \in G$ maps to $\alpha \in H_1(A)$ represented by the 1–cycle $(e_1, \ldots, e_n)$, then

$$q(g) = \prod_{i=1}^{n} \lambda(e_i) / \lambda(\overline{e}_i). \tag{3.12}$$

Recall that when G is non-elementary, all GBS trees for G have the same elliptic elements. Thus, in this case, q is well defined in terms of G alone.

If V is any non-trivial elliptic subgroup of G, then there is a formula

$$|q(g)| = [V : V \cap V^g]/[V^g : V \cap V^g] \tag{3.13}$$

(see [For06, Section 6]). Note that for $V = G_x$, the right hand side is the ratio of indices $i(\sigma)/i(\overline{\sigma})$ for the segment $\sigma = [x, gx]$.

3.6. **The orientation character.** Let G be a GBS group with GBS tree X. The *orientation character* of G (which a priori depends on X) is a homomorphism $\chi\colon G \to \{\pm 1\}$ defined by

$$\chi(g) = q(g)/|q(g)|.$$

When G is non-elementary χ is well defined in terms of G alone, since this is true of q. When G is $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, or $\bigcup_i C_i$, every orientation character is trivial. In the case of the Klein bottle group, there are two deformation spaces of GBS trees, one with trivial and one with non-trivial orientation character. Note that every GBS group has a subgroup of index at most 2 with trivial orientation character.

Remark 3.14. Let G be a GBS group with GBS tree X and quotient labeled graph (A, λ). If the orientation character is trivial then the label function λ can be made positive by *admissible sign changes*; see [CF08, Lemma 2.7].

An admissible sign change is the change in λ that results from changing the choices of generators in the graph of groups defined by (A, λ). The graph of groups itself does not change, only its description in terms of labels. If one changes the generator of a vertex group G_v, then the labels of edges in $E_0(v)$ all change sign. If one changes the generator of an edge group G_e, then $\lambda(e)$ and $\lambda(\overline{e})$ both change sign.

3.7. **Labeled graphs and deformations.** It often happens that different labeled graphs define isomorphic GBS groups. A given GBS group generally has no preferred GBS tree (or labeled graph). Indeed, it is still an open problem to find an algorithm which determines whether two labeled graphs define the same group. (This problem has been solved in some special cases; see especially [For06, Lev07, CF08, Dud17, CRKZ21].)

It is true, however, that any two cocompact GBS trees for a non-elementary G are related by an *elementary deformation* [For02]. This means that they are related by a finite sequence of elementary moves, called elementary collapses and expansions.

Definition 3.15. Let X be a G–tree and e an edge of X with endpoints in distinct orbits, such that $G_e = G_{\partial_0(e)}$. One obtains a new G–tree Y by collapsing each component

of the subforest spanned by Ge to a vertex. This operation is called an *elementary collapse* move. The vertex which is the image of e has stabilizer $G_{\partial_1(e)}$ and the set of elliptic subgroups of G does not change.

In the setting of GBS trees it is convenient to use a slightly restricted definition of expansion move (compared to [For02], where it is just the reverse of a collapse move). Let X and Y be as above. The transition from Y to X is called an *elementary expansion* move in all cases *except* when the vertex $\partial_0(e)$ has valence one and has trivial stabilizer.

The effect of an elementary move on the quotient graph of groups is as follows (with $A = G_{\partial_1(e)}$ and $C = G_e = G_{\partial_0(e)}$):
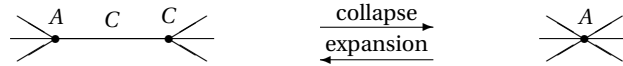

The elementary deformation shown below is called a *slide move*. To perform the move it is required that $D \subseteq C$. The edge with label C is allowed be a loop.
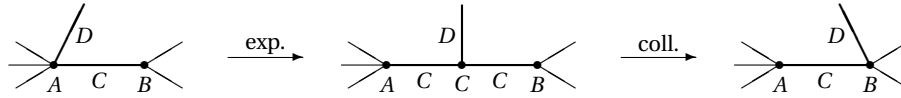

With the definitions given above, any elementary move performed on a GBS tree results in a GBS tree. The next proposition gives a description of these moves in terms of labeled graphs.
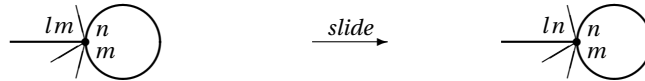
Proposition 3.16 (Prop. 2.4 of [For06])**.** *If an elementary move is performed on a generalized Baumslag–Solitar tree, then the quotient graph of groups changes locally as follows:*
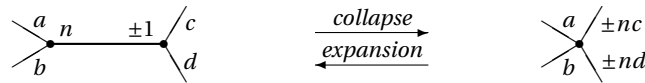
A slide move has the following description:
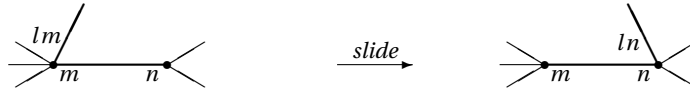
or

The moves above may be interpreted topologically, as homotopy equivalences between fibered 2–complexes in which annuli are collapsed or expanded or slid over one another.

Remark 3.17. The claim that any two cocompact GBS trees for a non-elementary G are related by an elementary deformation remains true with our restricted notion of expansion move. The proofs given in [For02] and [For06] produce deformations that do not use the forbidden expansion move, since they come from folds, which can only increase stabilizers. Alternatively, one may make both trees reduced and then apply the main result of [CF09], which produces deformations of a restricted kind in which all trees are minimal. (The forbidden move creates a non-minimal tree.)

3.8. **Subgroups of GBS groups.** Every subgroup of a GBS group is either a GBS group or a free group. In the former case the inclusion $H \hookrightarrow G$ is induced by a covering map from one fibered 2–complex to another. Such covering maps are encoded by *admissible branched coverings* of labeled graphs, defined as follows (cf. [Lev15, Lemma 5.3]).

Definition 3.18. An *admissible branched covering* of labeled graphs, from (A, λ) to (B, μ), consists of a surjective graph morphism $p \colon A \to B$ between connected graphs together with a *degree* function

$$d \colon E(A) \sqcup V(A) \to \mathbb{N}$$

satisfying $d(e) = d(\overline{e})$ for all $e \in E(A)$ such that the following holds. Given an edge $e \in E(B)$ with $\partial_0(e) = v$ and a vertex $u \in p^{-1}(v)$, let $k_{u,e} = \gcd(d(u), \mu(e))$. Then

(1) $|p^{-1}(e) \cap E_0(u)| = k_{u,e}$
(2) $\lambda(e') = \mu(e)/k_{u,e}$ for each edge $e' \in p^{-1}(e) \cap E_0(u)$
(3) $d(e') = d(u)/k_{u,e}$ for each edge $e' \in p^{-1}(e) \cap E_0(u)$.
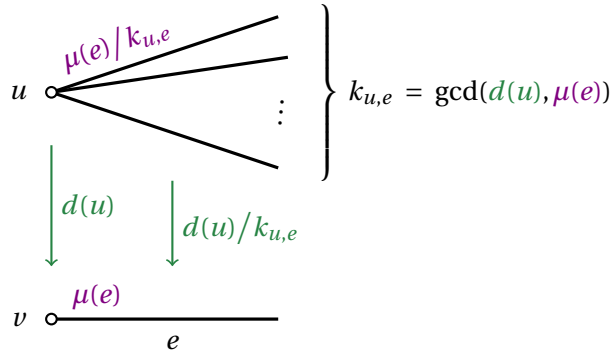
See Figure 1.

FIGURE 1. The admissibility condition. Each edge of $p^{-1}(e) \cap E_0(u)$ has label $\mu(e)/k_{u,e}$ and degree $d(u)/k_{u,e}$. There are $k_{u,e}$ such edges.

By lifting the graph of spaces structure, every covering space of a fibered 2–complex has a compatible graph of spaces structure, where either every vertex and edge space is

a line, or every vertex and edge space is a circle. The two cases correspond to whether
the subgroup acts freely on the Bass–Serre tree or is a GBS group (acting elliptically if it is
cyclic); see [For03, Lemmas 2.6–2.7]. Note that there is an induced surjective morphism
$p\colon A \to B$ of underlying graphs, and the covering map is fiber-preserving, with p the
induced map on leaf spaces.

Proposition 3.19. *Let G be a GBS group with labeled graph (B,μ). There is a one-to-one
correspondence between conjugacy classes of GBS subgroups of G (excluding hyperbolic
cyclic subgroups) and admissible branched coverings $(A,\lambda) \to (B,\mu)$.*

This result is very similar to [Lev15, Lemma 6.3], but we explain it here somewhat
differently.

Proof. It suffices to classify the fibered 2–complex covering spaces of $Z_{(B,\mu)}$. Recall that
$Z_{(B,\mu)}$ is a union of mapping cylinders M_e. The admissibility condition is simply a de-
scription of the finite-sheeted covers of these subspaces M_e. As a mapping cylinder, M_e
deformation retracts onto $C_{\partial_0(e)}$, and has infinite cyclic fundamental group. However it
is also a fiber bundle over this circle, giving it the structure of a mapping *torus*. The fiber
is a cone on $n = |\mu(e)|$ points, denoted by $C(n)$, and the monodromy $\varphi\colon C(n) \to C(n)$ is
the automorphism which permutes the n points by an n–cycle σ. Let $z_1,\ldots,z_n$ be the n
points and c the cone point. Writing M_e as

$$C(n) \times [0,1]\big/(x,0) \sim (\varphi(x),1),$$

the image of $c \times [0,1]$ is the singular circle $C_{\partial_0(e)}$ and the images of $z_i \times [0,1]$ join up to
form the circle C_e. The map $C_e \to C_{\partial_0(e)}$ in which each $z_i \times \{t\}$ maps to $c \times \{t\}$ is the degree
$\mu(e)$ covering map defining M_e.

For any $d \in \mathbb{N}$, the d–sheeted covering space N of M_e is the mapping torus of

$$\varphi^d\colon C(n) \to C(n).$$

The permutation σ^d decomposes into $k = \gcd(d,n)$ disjoint (n/k)–cycles. Thus the pre-
image of C_e in N is k disjoint circles and N is the mapping cylinder of the map $\bigsqcup_{i=1}^k S^1 \to$
S^1 in which each component maps by degree $\mu(e)/k$. Each of the circles above C_e covers
with degree d/k, and the circle above $C_{\partial_0(e)}$ covers with degree d. Now N is exactly
described by the admissibility condition as shown in Figure 1, with $d = d(u)$. Thus for
any covering map of fibered 2–complexes, the induced morphism of indexed graphs is
an admissible branched covering. The degree function records the degrees of each fiber
in the cover mapping to its image. (Orientations of the fibers in the cover are chosen so
that these degrees are all positive.)

Conversely, let $p\colon (A,\lambda) \to (B,\mu)$ be an admissible branched covering. Recall that $Z_{(A,\lambda)}$ and $Z_{(B,\mu)}$ are, respectively, unions of the subspaces $M_{e'}$ ($e' \in E(A)$) and M_e ($e \in E(B)$). For each $e \in E(B)$ with $\partial_0(e) = v$ and each $u \in p^{-1}(v)$ the subspace

$$N_{u,e} = \bigcup_{e' \in p^{-1}(e) \cap E_0(u)} M_{e'}$$

of $Z_{(A,\lambda)}$ has a degree $d(u)$ covering map $p_{u,e}\colon N_{u,e} \to M_e$, by the admissibility condition. These subspaces $N_{u,e}$ intersect each other only along the circles C_u and $C_{e'}$. The restrictions $p_{u,e}|_{C_u}$ and $p_{u,e}|_{C_{e'}}$ are coverings of degrees $d(u)$ and $d(e')$ respectively. By adjusting the covering maps on $N_{u,e}$ by fiber-preserving isotopies, the restrictions of the coverings to the circles C_u and $C_{e'}$ (for $u \in V(A)$, $e' \in E(A)$) can be made to all agree on each circle. Then the covering maps $p_{u,e}$ join to give a covering of fibered 2–complexes. $\square$

Remark 3.20. Every *finite index* subgroup of G is a GBS subgroup, and these correspond to the branched coverings $(A,\lambda) \to (B,\mu)$ for which the morphism $A \to B$ has finite fibers. More generally, if $H < G$ corresponds to the branched covering $p\colon (A,\lambda) \to (B,\mu)$, then $[G:H] = \sum_{u \in p^{-1}(v)} d(u)$, for any vertex v of B.

Remark 3.21. Every GBS group is *coherent*, meaning that every finitely generated subgroup is finitely presented. Since every subgroup is free or GBS, it suffices to note that a GBS group G with minimal GBS tree X is finitely generated if and only if X is cocompact. In that case, G is the fundamental group of a compact fibered 2–complex, and is finitely presented.

4. The 2–complexes $X_{m,n}$ and $X_{(A,\lambda)}$

Fix positive integers m, n and let $k = \gcd(m, n)$. Let $T_{m,n}$ denote the directed simplicial tree in which every vertex has m incoming edges and n outgoing edges. This tree is the Bass–Serre tree of $BS(m, n)$ with its standard labeled graph. The orientations are such that the stable letter t in the presentation (3.4) is directed forward, and has axis in $T_{m,n}$ that is a directed line which t shifts forward one unit.

Recall from Example 3.3 the fibered 2–complex $Z_{m,n}$. It is homeomorphic to the presentation 2–complex for the presentation of $BS(m, n)$ given in (3.4). That CW complex has one vertex, two edges labeled a and t, and one 2–cell attached by the boundary word $ta^m t^{-1} a^{-n}$. The cell structure we want to put on $Z_{m,n}$ is obtained from this by subdivision and is illustrated in Figure 2. The initial 2–cell has been split into k 2–cells, with new 1–cells labeled $t_1, \ldots, t_{k-1}$. The new 2–cells are attached by the words $t_i a^{m/k} t_{i+1}^{-1} a^{-n/k}$ (indices modulo k), with t_0 understood to mean t. There is still only one vertex.

The 2–complex $X_{m,n}$ is defined to the the universal cover of $Z_{m,n}$ with the induced cell structure. It is tiled entirely by quadrilateral 2–cells of the kind shown in Figure 2, with
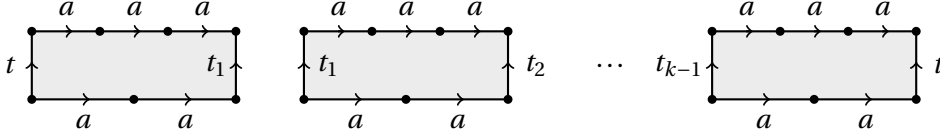
FIGURE 2. The cell structure for $Z_{m,n}$ when $m = 3k$, $n = 2k$.

sides of length $1, m/k, 1, n/k$. The reason for subdividing the initial cell structure of $Z_{m,n}$ is to increase the homogeneity of $X_{m,n}$ and make it as symmetric as possible.

We extend the definition to allow $m, n \in \mathbb{Z} - \{0\}$ by declaring $X_{m,n} = X_{|m|,|n|}$. Then $BS(m, n)$ is a lattice in $\mathrm{Aut}(X_{m,n})$ for any m, n (since $X_{m,n}$ is the universal cover of $Z_{\pm m, \pm n}$). However, when discussing an unspecified $X_{m,n}$, the default assumption will be that $m, n > 0$.

4.1. **Combinatorial description.**

The space $X_{m,n}$ is homeomorphic to $T_{m,n} \times \mathbb{R}$, but combinatorially and geometrically it is very far from being a product (unless $m = n$). Let $\pi \colon X_{m,n} \to T_{m,n}$ be the projection map. The 1–cells of $X_{m,n}$ will be called *horizontal* if they map to a vertex of $T_{m,n}$ and *vertical* if they map to an edge. The vertical edges inherit orientations from the edges of $T_{m,n}$, consistent with the orientations labeled t or t_i in Figure 2. We view this direction as the "upward" direction in $X_{m,n}$ and in $T_{m,n}$. (The horizontal 1–cells are *not* oriented; the orientations labeled a in Figure 2 should be ignored when considered as cells in $X_{m,n}$.)

For any $v \in V(T_{m,n})$ the pre-image $\pi^{-1}(v)$ will be called a *branching line*. The pre-image of a closed edge of $T_{m,n}$ will be called a *strip*. Note that the branching lines are tiled by horizontal edges. We define an (i, j)–*cell* to be a 2–cell attached by a combinatorial path consisting of one upward vertical edge, i horizontal edges, one downward vertical edge, and j horizontal edges. An (i, j)–*strip* is a bi-infinite sequence of (i, j)–cells, joined along their vertical edges. Every strip in $X_{m,n}$ is a $(m/k, n/k)$–strip.

We may regard $X_{m,n}$ as being assembled from branching lines and $(m/k, n/k)$–strips just as $T_{m,n}$ is made of vertices and edges. Each branching line has n strips above it and m strips below it. When attaching a strip *above* a branching line, there are n/k ways to do this; if we identify the vertices of the branching line with $\mathbb{Z}$, the vertical edges of the strip will meet a coset $i + (n/k)\mathbb{Z}$ for some i. In $X_{m,n}$, the n strips are joined along the cosets $i + (n/k)\mathbb{Z}$ for $i = 1, \ldots, n$. Thus every vertex on the line has k outgoing vertical edges.

The strips below the branching line are attached in a similar manner; there are m of them, attached along the cosets $i + (m/k)\mathbb{Z}$ for $i = 1, \ldots m$. Each vertex in the line has k

incoming vertical edges. This description of the neighborhood of every branching line, together with the projection to $T_{m,n}$, completely determines $X_{m,n}$ as a CW complex.

4.2. **Metric structure.** The complex $X_{m,n}$ admits a piecewise-Riemannian metric on which $\mathrm{Aut}(X_{m,n})$ acts by isometries. The construction is based on [FM98], which treats the case of $X_{1,n}$. If $m = n$ then each 2–cell is isometric to a Euclidean $m/k \times 1 = 1 \times 1$ rectangle, and all 1–cells have length 1. If $m \neq n$ then each 2–cell is isometric to a right-angled quadrilateral region in the hyperbolic plane (a "horobrick") whose vertical sides are geodesics of length $|\log(m/n)|$ and whose horizontal sides are concentric horocyclic arcs of lengths m/k and n/k. This metric gives every horizontal 1–cell length 1.

When $m \neq n$, each strip is isometric to the region in $\mathbb{H}^2$ between two concentric horocycles of distance $|\log(m/n)|$ apart. Since the branching lines are horocycles, $X_{m,n}$ has concentrated positive curvature along these lines, and is definitely not hyperbolic in any global sense. On the other hand, each strip in $X_{m,m}$ is isometric to a Euclidean strip of width 1, and $X_{m,m}$ is isometric to $T_{m,m} \times \mathbb{R}$.

4.3. **Projection.** The complex $X_{1,1}$ is the Euclidean plane tiled by unit squares. It admits rotations and the branching lines are not invariant. In all other cases, branching lines and strips in $X_{m,n}$ are preserved by $\mathrm{Aut}(X_{m,n})$. Hence the projection $\pi\colon X_{m,n} \to T_{m,n}$ induces a homomorphism $\pi_*\colon \mathrm{Aut}(X_{m,n}) \to \mathrm{Aut}(T_{m,n})$. Now choose consistent orientations for all of the branching lines. Every $g \in \mathrm{Aut}(X_{m,n})$ either preserves the orientations of all branching lines, or reverses all of them. We will call g *orientation preserving* or *orientation reversing* accordingly.

Lemma 4.1. *Suppose $m < n$. Let α be a directed ray in $T_{m,n}$ and suppose $\pi_*(g)$ fixes α pointwise, for some orientation preserving $g \in \mathrm{Aut}(X_{m,n})$. Then g fixes $\pi^{-1}(\alpha)$ pointwise.*

A similar statement holds for "anti-directed" rays if $m > n$.

Proof. The metric on $\pi^{-1}(\alpha)$ is isometric to a closed horoball in $\mathbb{H}^2$. Thus g acts by a hyperbolic isometry preserving this horoball. In the upper half plane model with the center of the horoball at infinity, the only such isometries are reflections about vertical lines and horizontal translations. Since g is orientation preserving, it is a translation. However, in this model, the 2–cells are rectangular regions of the form $[a, b] \times [c, d]$ with width $|b - a|$ getting arbitrarily large as one moves upward in the plane. No horizontal translation can preserve such a cell structure, except for the identity. $\square$

Corollary 4.2. *If $m \neq n$ then $\pi_*\colon \mathrm{Aut}(X_{m,n}) \to \mathrm{Aut}(T_{m,n})$ is continuous and injective.*

Proof. For continuity note that the pre-image of $U^{T_{m,n}}_{\sigma \to \tau}$ is the union of the sets $U^{X_{m,n}}_{\tilde{\sigma} \to \tilde{\tau}}$ for $\tilde{\sigma} \in \pi^{-1}(\sigma)$, $\tilde{\tau} \in \pi^{-1}(\tau)$. For injectivity, suppose first that $g \in \ker(\pi_*)$ is orientation

preserving. Then $g = $ id by Lemma 4.1, since $T_{m,n}$ is a union of rays of the relevant type (directed or anti-directed).

If g is orientation reversing, then it acts on every branching line as a reflection with a unique fixed point, and similarly on every strip as a reflection. In the latter case, the line segment of fixed points is either a vertical edge or it passes through the center of a 2–cell. If, say, $m < n$ and $n/k > 2$, consider two strips whose lower sides are the same branching line, with vertical edges joined along cosets one unit apart. Then the fixed point sets of the strips cannot agree on the branching line and we have a contradiction. If $n/k = 2, m/k = 1$ then one finds a similar contradiction by considering an arrangement of four strips whose projection to $T_{m,n}$ is a segment of two downward edges followed by two upward edges. $\qquad\square$

Remark 4.3. The map π_* is not an embedding. The topology on $\mathrm{Aut}(X_{m,n})$ is strictly finer than the subspace topology on $\pi_*(\mathrm{Aut}(X_{m,n}))$. One can show that $\pi_*(U^{X_{m,n}}_{\sigma \to \tau})$ is not open in $\pi_*(\mathrm{Aut}(X_{m,n}))$, for any cells σ, τ with $U^{X_{m,n}}_{\sigma \to \tau} \neq \emptyset$.

Proof in the case $\sigma = \tau$. Note that $\pi_*(U^X_{\sigma \to \sigma})$ contains the identity element of $\mathrm{Aut}(T) \cap \pi_*(\mathrm{Aut}(X))$. Every basic neighborhood of the identity has the form $U^T_{\sigma_1 \to \sigma_1} \cap \cdots \cap U^T_{\sigma_\ell \to \sigma_\ell} \cap \pi_*(\mathrm{Aut}(X))$. Let $S \subset T$ be a finite subtree containing $\pi(\sigma), \sigma_1, \ldots, \sigma_\ell$. The subcomplex $\pi^{-1}(S)$ admits a non-trivial shift which projects to the identity on S. This shift extends to an automorphism $g \in \mathrm{Aut}(X_{m,n})$. Now $\pi_*(g) \in U^T_{\sigma_1 \to \sigma_1} \cap \cdots \cap U^T_{\sigma_\ell \to \sigma_\ell} \cap \pi_*(\mathrm{Aut}(X))$ while $g \notin U^X_{\sigma \to \sigma}$. Since π_* is injective, $\pi_*(g) \notin \pi_*(U^X_{\sigma \to \sigma})$. Hence no basic neighborhood of the identity is contained in $\pi_*(U^X_{\sigma \to \sigma})$. $\qquad\square$

4.4. **General labeled graphs.** For any labeled graph (A, λ) let $T_{(A,\lambda)}$ denote the corresponding Bass–Serre tree. We will define $X_{(A,\lambda)}$ to be the universal cover of $Z_{(A,\lambda)}$ with a suitable cell structure.

For each v put a cell structure on the circle C_v with one vertex and one edge. Given $e \in E(A)$, let $n_e = |\lambda(e)|$, $m_e = |\lambda(\overline{e})|$, and $k_e = \gcd(m_e, n_e)$. The annulus $M_e \cup M_{\overline{e}}$ has boundary curves of lengths n_e and m_e attached to $C_{\partial_0(e)}$ and $C_{\partial_1(e)}$ respectively. Tile this annulus with k_e $(m_e/k_e, n_e/k_e)$–cells. Doing this for every edge we obtain a cell structure for $Z_{(A,\lambda)}$, which then induces one on $X_{(A,\lambda)}$.

Every strip above the annulus $M_e \cup M_{\overline{e}}$ is an $(m_e/k_e, n_e/k_e)$–strip, and $X_{(A,\lambda)}$ is homeomorphic to $T_{(A,\lambda)} \times \mathbb{R}$. For each e, every branching line covering $C_{\partial_0(e)}$ has n_e strips above it covering $M_e \cup M_{\overline{e}}$, attached along the cosets $i + (n_e/k_e)\mathbb{Z}$ for $i = 1, \ldots, n_e$. Every branching line covering $\partial_1(e)$ has m_e strips below it covering $M_e \cup M_{\overline{e}}$, attached along the cosets $i + (m_e/k_e)\mathbb{Z}$ for $i = 1, \ldots, m_e$.

The space $X_{(A,\lambda)}$ admits an $\mathrm{Aut}(X_{(A,\lambda)})$–invariant metric using the construction from Section 4.2. Each $(m_e/k_e, n_e/k_e)$–cell is metrized as a horobrick of height $|\log(m_e/n_e)|$

with horocyclic sides of lengths m_e/k_e and n_e/k_e (if $m_e \neq n_e$), or as a Euclidean 1×1 square (if $m_e = n_e$). The $(m_e/k_e, n_e/k_e)$–strips are then horocyclic strips or Euclidean strips accordingly. This metric on $X_{(A,\lambda)}$ is quasi-isometric, but not isometric, to the one used by Whyte in [Why01]. In the latter paper, each (m, n)–horocyclic strip has width 1 and constant curvature $-|\log(m/n)|$, whereas here it has width $|\log(m/n)|$ and constant curvature -1.

Remark 4.4. The notions of *orientation preserving* and *reversing* automorphisms of $X_{m,n}$ apply equally well to $X_{(A,\lambda)}$ whenever $X_{(A,\lambda)} \ncong X_{1,1}$. If G is the GBS group defined by (A, λ), then these notions agree with the orientation character on G. That is, $g \in G <$ $\mathrm{Aut}(X_{(A,\lambda)})$ is orientation preserving if and only if $\chi(g) = 1$.

In this way, the orientation character extends to a homomorphism $\chi \colon \mathrm{Aut}(X_{(A,\lambda)}) \to \{\pm 1\}$, even if the modular homomorphism q does not.

4.5. **Torsion-free lattices in** $X_{m,n}$**.** One simple way for a GBS group to be a lattice in $\mathrm{Aut}(X_{m,n})$ is if its labeled graph (A, λ) satisfies $X_{(A,\lambda)} \cong X_{m,n}$. The next result gives a criterion for this.

Proposition 4.5. *Let G be the GBS group defined by (A, λ) and suppose there is a directed graph structure $E(A) = E^+(A) \sqcup E^-(A)$ on A such that*

(1) *for every $v \in V(A)$,*

$$\sum_{e \in E_0^+(v)} |\lambda(e)| = n \ \text{ and } \ \sum_{e \in E_0^-(v)} |\lambda(e)| = m$$

(2) *for every $e \in E^+(A)$, let $n_e = |\lambda(e)|$, $m_e = |\lambda(\overline{e})|$, $k_e = \gcd(m_e, n_e)$, and $k = \gcd(m, n)$; then*

$$n_e/k_e = n/k \ \text{ and } \ m_e/k_e = m/k.$$

Then $X_{(A,\lambda)} \cong X_{m,n}$, and hence G is a lattice in $\mathrm{Aut}(X_{m,n})$.

Proof. Condition (1) says that the tree $T_{(A,\lambda)}$, with directed graph structure induced from A, is isomorphic to $T_{m,n}$. This is so because the two sums count the numbers of strips entering (resp. exiting) each branching line in $X_{(A,\lambda)}$. Condition (2) says that every strip in $X_{(A,\lambda)}$ is a $(m/k, n/k)$–strip. It remains to examine how these strips join the branching lines.

Fix a vertex $v \in V(A)$ and a branching line L covering C_v. For each $e \in E^+(A) \cap E_0(v)$ there are $|\lambda(e)|$ strips above L mapping to $M_e \cup M_{\overline{e}}$. These are attached along the cosets $i + (n/k)\mathbb{Z}$ for $i = 1, \ldots, |\lambda(e)|$. Each coset has the same number of such strips attached to it (namely, k_e). Hence, overall, the n strips above L are distributed evenly among the cosets of $(n/k)\mathbb{Z}$, with k of them attached along each one. The strips attached below L

are also equidistributed among the cosets of $(m/k)\mathbb{Z}$. Now $X_{(A,\lambda)}$ matches the description of $X_{m,n}$ from Section 4.1. $\hfill\square$

Now suppose $m \neq n$ and consider a general torsion-free uniform lattice G in $\mathrm{Aut}(X_{m,n})$. It acts freely and cocompactly on $X_{m,n}$ by Proposition 2.1. Note that every automorphism preserves the directed structure on $T_{m,n}$ because $m \neq n$. In particular, no strip has its sides exchanged, so every strip covers an annulus in $X_{m,n}/G$. Each branching line covers a circle. The quotient is then a compact fibered 2–complex, homeomorphic to some $Z_{(A,\lambda)}$. Moreover the graph A is a directed graph, with directed structure induced by that of $T_{m,n}$.

We put a cell structure on $Z_{(A,\lambda)}$ by identifying it with $X_{m,n}/G$. There is a *length function* $\ell\colon V(A) \sqcup E(A) \to \mathbb{N}$ defined as follows. For $v \in V(A)$, $\ell(v)$ is the combinatorial length of the circle C_v. For $e \in E(A)$, $\ell(e)$ is the number of 2–cells tiling the annulus $M_e \cup M_{\overline{e}}$ (that is, its combinatorial girth). Note that $\ell(e) = \ell(\overline{e})$ for all $e \in E(A)$. The edges in $M_e \cup M_{\overline{e}}$ crossing from one boundary component to the other are called *vertical edges*, since they are the images of vertical edges of $X_{m,n}$. They are directed, consistently with e.

Proposition 4.6. *Suppose $m \neq n$ and let G be a torsion-free uniform lattice in $\mathrm{Aut}(X_{m,n})$. Let $k = \gcd(m,n)$, $m' = m/k$, and $n' = n/k$. Let (A,λ) be a directed labeled graph such that $X_{m,n}/G$ is homeomorphic to $Z_{(A,\lambda)}$, with associated length function $\ell\colon V(A) \sqcup E(A) \to \mathbb{N}$ and directed structure induced from $T_{m,n}$. Then*

(1) *for every $v \in V(A)$,*

$$\sum_{e\in E_0^+(v)} |\lambda(e)| = n \ \ and \ \ \sum_{e\in E_0^-(v)} |\lambda(e)| = m$$

(2) *for every $e \in E^+(A)$,*

$$\ell(\partial_0(e))|\lambda(e)| = n'\ell(e),$$
$$\ell(\partial_1(e))|\lambda(\overline{e})| = m'\ell(e)$$

(3) *for every $v \in V(A)$, let $k_0(v) = \gcd(\ell(v), n')$ and $k_1(v) = \gcd(\ell(v), m')$; then there exist partitions*

$$E_0^+(v) = E_1^+ \sqcup \cdots \sqcup E_{k_0(v)}^+, \quad E_0^-(v) = E_1^- \sqcup \cdots \sqcup E_{k_1(v)}^-$$

such that the sums $\sum_{e\in E_i^+} |\lambda(e)|$ are equal for all i, and the sums $\sum_{e\in E_j^-} |\lambda(e)|$ are equal for all j.

(4) *for every $v \in V(A)$,*

$$\sum_{e \in E_0^+(v)} \ell(e) \; = \; k\ell(v) \; = \; \sum_{e \in E_0^-(v)} \ell(e)$$

(5) *the directed graph A is strongly connected.*

Proof. Conclusion (1) follows exactly as in Proposition 4.5. Conclusion (2) is evident from the cell structure on the annulus $M_e \cup M_{\bar{e}}$, which is tiled by (m', n')–cells. Its boundary curves have lengths $n'\ell(e)$ and $m'\ell(e)$, and they wrap $|\lambda(e)|$ times and $|\lambda(\bar{e})|$ times respectively around $C_{\partial_0(e)}$ and $C_{\partial_1(e)}$.

For (3), identify the vertices of C_v with the cyclic group $\mathbb{Z}/\ell(v)\mathbb{Z}$ (in their natural cyclic ordering). The element $n' + \ell(v)\mathbb{Z}$ generates the subgroup $k_0(v)\mathbb{Z}/\ell(v)\mathbb{Z}$. Given $e \in E_0^+(v)$ the annulus $M_e \cup M_{\bar{e}}$ has $\ell(e)$ outgoing vertical edges incident to vertices of C_v, spaced n' units apart. They meet the vertices along a coset of $k_0(v)\mathbb{Z}/\ell(v)\mathbb{Z}$ in $\mathbb{Z}/\ell(v)\mathbb{Z}$, with $\ell(e)k_0(v)/\ell(v)$ outgoing edges incident to each such vertex.

For each $i = 1, \dots, k_0(v)$ let E_i^+ be the set of edges $e \in E_0^+(v)$ such that the vertical edges of $M_e \cup M_{\bar{e}}$ are joined to C_v along the coset $i + (k_0(v)\mathbb{Z}/\ell(v)\mathbb{Z})$ in $\mathbb{Z}/\ell(v)\mathbb{Z}$. Now every vertex in the coset has

$$\sum_{e \in E_i^+} \ell(e)k_0(v)/\ell(v) \; \overset{(2)}{=} \; \sum_{e \in E_i^+} |\lambda(e)|k_0(v)/n' \; = \; (k_0(v)/n') \sum_{e \in E_i^+} |\lambda(e)|$$

outgoing vertical edges incident to it. In the universal cover $X_{m,n}$, every vertex has the same number of outgoing vertical edges; hence the same is true of $X_{m,n}/G$ and the sums $\sum_{e \in E_i^+} |\lambda(e)|$ are the same for all i. The statement for $E_0^-(v)$ is proved similarly.

Conclusion (4) follows from (1) and (2):

$$\sum_{e \in E_0^+(v)} \ell(e) \; \overset{(2)}{=} \; \sum_{e \in E_0^+(v)} \ell(v)|\lambda(e)|/n' \; \overset{(1)}{=} \; \ell(v)n/n' \; = \; k\ell(v)$$

and similarly for the second sum.

For (5), define a new directed graph A' from A by replacing each directed edge e with $\ell(e)$ directed edges. By (4), each vertex of A' has equal numbers of incoming and outgoing edges. It follows that $E^+(A')$ admits a partition into directed cycles. Hence every directed edge is part of a directed cycle. The same is then true of A. Now consider the decomposition of A into strongly connected components. If this decomposition is nontrivial then there is a directed edge between two such components that cannot be part of a directed cycle, which is a contradiction. Hence A is strongly connected. $\qquad\square$

Theorem 4.7. *Suppose $m \neq n$ and let G be a torsion-free group. Then G is isomorphic to a uniform lattice in $\mathrm{Aut}(X_{m,n})$ if and only if there exist a compact GBS structure (A, λ) for G, a directed graph structure $E(A) = E^+(A) \sqcup E^-(A)$, and a function $\ell : V(A) \cup E(A) \to \mathbb{N}$*

satisfying $\ell(e) = \ell(\overline{e})$ for all $e \in E(A)$ such that conclusions (1), (2), and (3) of Proposition 4.6 hold.

Proof. The "only if" direction holds by Proposition 4.6. For the converse let (A, λ), the directed graph structure, and ℓ be given. Let $k = \gcd(m, n)$, $m' = m/k$, and $n' = n/k$. We put a cell stucture on $Z_{(A,\lambda)}$ as follows. For each $v \in V(A)$ let C_v have $\ell(v)$ vertices and $\ell(v)$ edges. Choose an identification of the cyclically ordered vertices with $\mathbb{Z}/\ell(v)\mathbb{Z}$. For each $e \in E^+(A)$ let A_e be an annulus tiled by $\ell(e)$ (m', n')-cells, so that the initial end is a circle of length $n'\ell(e)$ and the opposite end has length $m'\ell(e)$. The $\ell(e)$ edges joining the two ends are called vertical edges, and are directed from the initial end to the other end.

For each v and $e \in E_0^+(v)$ there are $k_0(v)$ ways, combinatorially, to attach the initial end of A_e to C_v by a degree $\lambda(e)$ map, corresponding to the coset of $k_0(v)\mathbb{Z}/\ell(v)\mathbb{Z}$ in $\mathbb{Z}/\ell(v)\mathbb{Z}$ met by the vertical edges of A_e. (The existence of these combinatorial attaching maps follows from (2).) Using the partition $E_0^+(v) = E_1^+ \sqcup \cdots \sqcup E_{k_0(v)}^+$, attach each annulus A_e along the coset $i + (k_0(v)\mathbb{Z}/\ell(v)\mathbb{Z})$ such that $e \in E_i^+$. Attach the other ends of each annulus similarly along the cosets of $k_1(v)\mathbb{Z}/\ell(v)\mathbb{Z}$. The resulting space is homeomorphic to $Z_{(A,\lambda)}$.

By (1) we have $T_{(A,\lambda)} \cong T_{m,n}$ as directed graphs. By construction, each strip in the universal cover of $Z_{(A,\lambda)}$ is an (m', n')–strip. Finally, by (3), each vertex of $Z_{(A,\lambda)}$ has equal numbers of outgoing (respectively, incoming) vertical edges. It follows that the strips in the universal cover are joined to the branching lines in the manner described in Section 4.1. Hence the universal cover of $Z_{(A,\lambda)}$ is isomorphic to $X_{m,n}$ as a CW complex, and G acts freely and cocompactly on $X_{m,n}$. $\square$

4.6. **Discreteness.** One clear situation in which $\mathrm{Aut}(X_{m,n})$ cannot contain incommensurable lattices is when it is discrete. When this occurs, every lattice has finite index in $\mathrm{Aut}(X_{m,n})$, by [BL01, 1.7].

Theorem 4.8. *For any $m, n \geq 1$ the group $\mathrm{Aut}(X_{m,n})$ is discrete if and only if $\gcd(m, n) = 1$. When this occurs, there is a short exact sequence*

$$1 \to BS(m, n) \to \mathrm{Aut}(X_{m,n}) \xrightarrow{\chi} \{\pm 1\} \to 1.$$

Proof. Suppose $\gcd(m, n) = 1$ and let K be the kernel of $\chi \colon \mathrm{Aut}(X_{m,n}) \to \{\pm 1\}$. We claim that K acts freely on the vertices of $X_{m,n}$. If $g \in K$ fixes a vertex $x \in X_{m,n}$ then it also fixes pointwise the branching line L containing x. Now note from Section 4.1 that the strips above this branching line are all attached along different cosets $i + n\mathbb{Z}$, and the same is true of the strips below L. Hence g cannot permute these strips and it acts trivially on

them. By similar reasoning, g acts trivially on the radius r neighborhood of L, for every r, so $g = 1$.

The lattice $BS(m, n)$ is a subgroup of K acting transitively on the vertices of $X_{m,n}$. Since K acts freely on these vertices, we must have $K = BS(m, n)$.

Now suppose that $\gcd(m, n) = k \neq 1$. Let $K = \ker(\chi)$ and consider a vertex stabilizer K_x. Choose any directed ray α in $T_{m,n}$ with initial vertex $\pi(x)$. Let L_x be the branching line through x and for any vertex $v \in \alpha$ let L_v be the branching line $\pi^{-1}(v)$. There are k strips attached above L_v along each coset $i + (n/k)\mathbb{Z}$. In particular there are elements $g \in K_x$ which fix L_x and L_v pointwise, but permute the strips above L_v nontrivially. Since v is arbitrarily far from x, K_x is infinite and K is non-discrete. (With a little more care one may express K_x explicitly as an inverse limit of products of permutation groups.) $\qquad\square$

5. A COMMENSURABILITY INVARIANT

Consider a non-elementary GBS group G and a non-trivial elliptic subgroup $V < G$. Recall that because G is non-elementary, the elliptic subgroups of G are well defined, as is the modular homomorphism $q\colon G \to \mathbb{Q}^\times$.

We define the *V-depth* of an element $g \in G$ to be $D_V(g) = [V : V \cap V^g]$. Next we define the *depth profile*:

$$\mathscr{D}(G, V) \;=\; \{D_V(g) \mid g \text{ is hyperbolic and } q(g) = \pm 1\} \subset \mathbb{N}.$$

The depth profile is a commensurability invariant in the following sense.

Theorem 5.1. *Let G be a non-elementary GBS group and $V < G$ a non-trivial elliptic subgroup.*

(1) *If $G' < G$ is a subgroup of finite index and $V \subset G'$ then*

$$\mathscr{D}(G, V) \;=\; \mathscr{D}(G', V).$$

(2) *If $V' < V$ with $[V : V'] = r$ then*

$$\mathscr{D}(G, V') \;=\; \{n/\gcd(r, n) \mid n \in \mathscr{D}(G, V)\}.$$

Once $\mathscr{D}(G, V)$ is known, using (2) one can compute depth profiles for every finite index subgroup of G, and hence for every GBS group commensurable with G, by (1). Alternatively, one may define an equivalence relation on the set of subsets of $\mathbb{N}$, by declaring $S \subset \mathbb{N}$ equivalent to $\{n/\gcd(r, n) \mid n \in S\}$ for each $r \in \mathbb{N}$ and taking the symmetric and transitive closure. Then the equivalence class of $\mathscr{D}(G, V)$ is a true commensurability invariant of G.

Given $S \subset \mathbb{N}$ and $r \in \mathbb{N}$ let S/r denote the set $\{n/\gcd(r, n) \mid n \in S\} \subset \mathbb{N}$.

Proposition 5.2. *Two subsets $S, S' \subset \mathbb{N}$ are equivalent if and only if there exist $r, r' \in \mathbb{N}$ such that $S/r = S'/r'$.*

Proof. Let $\mathrm{div}_r \colon \mathbb{N} \to \mathbb{N}$ denote the function $n \mapsto n/\gcd(r, n)$. One easily verifies that $\mathrm{div}_s \circ \mathrm{div}_r = \mathrm{div}_{rs}$, and therefore $(S/r)/s = S/(rs)$ for all $S \subset \mathbb{N}$ and $r, s \in \mathbb{N}$.

Let $\sim$ denote the smallest equivalence relation on $\mathscr{P}(\mathbb{N})$ such that $S \sim S/r$ for all $r \in \mathbb{N}$. Define $S \approx S'$ if there exist $r, r' \in \mathbb{N}$ such that $S/r = S'/r'$. Clearly, $S \approx S'$ implies $S \sim S'$. For the converse it suffices to show that $\approx$ is an equivalence relation, i.e. that it is transitive. If $S \approx S'$ and $S' \approx S''$ then there exist $r, r', s', s'' \in \mathbb{N}$ such that $S/r = S'/r'$ and $S'/s' = S''/s''$. Then

$$S/(rs') = (S/r)/s' = (S'/r')/s' = S'/(r's') = (S'/s')/r' = (S''/s'')/r' = S''/(s''r'),$$

so indeed $S \approx S''$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 5.3. Let $G = BS(k, kn)$ with $k, n > 1$. There is an index k normal subgroup $G' < G$ isomorphic to $\bigvee_{i=1}^{k} BS(1, n)$, with vertex group V. Since every edge in this decomposition of G' has indices 1 and n, every element of modulus ± 1 has V-depth a power of n. All powers are realized, and

$$\mathscr{D}(G, V) = \mathscr{D}(G', V) = \{n^i \mid i \in \mathbb{N} \cup \{0\}\}.$$

See Proposition 6.2 ahead for a more detailed proof (since by slide moves we can write $G' \cong BS(1, n) \vee \bigvee_{i=1}^{k-1} BS(1, 1)$). It is important that $k > 1$; otherwise there are no hyperbolic elements of modulus ± 1 and the depth profile of G' is empty.

Changing the elliptic subgroup to $V' < V$ with $[V, V'] = r$, Theorem 5.1(2) gives

$$\mathscr{D}(G, V') = \mathscr{D}(G, V)/r = \{n^i / \gcd(r, n^i) \mid i \in \mathbb{N} \cup \{0\}\}.$$

This set (for any r) has the property that, with finitely many exceptions, any two successive elements have ratio n. Hence the modulus n is an invariant of the equivalence class of depth profiles of $BS(k, kn)$. This yields a new proof of the most difficult case of [CRKZ21, Theorem 1.1]:

Corollary 5.4 (Lemma 7.2 of [CRKZ21])**.** *The groups $BS(k_1, k_1 n_1)$ and $BS(k_2, k_2 n_2)$ with $k_i, n_i > 1$ are commensurable only if $n_1 = n_2$.*

Theorem 5.1 will follow quickly from the next lemma.

Lemma 5.5. *Let G be a non-elementary GBS group and $V < G$ a non-trivial elliptic subgroup. Suppose g is hyperbolic and $q(g) = \pm 1$.*

    (1) $D_V(g) = D_V(g^k)$ *for all $k \geqslant 1$.*
    (2) *If $V' < V$ with $[V : V'] = r$ then $D_{V'}(g) = D_V(g)/\gcd(r, D_V(g))$.*

Proof. First we prove (2). Let $d = D_V(g)$ and let $V = \langle x \rangle$, $V^g = \langle y \rangle$. Then $V \cap V^g$ is the subgroup $\langle x^d \rangle$. Since $q(g) = \pm 1$ we have $[V^g : V \cap V^g] = d$ and so $V \cap V^g = \langle y^d \rangle$ also. We wish to identify the subgroup $V' \cap V'^g$, given that $V' = \langle x^r \rangle$ and $V'^g = \langle y^r \rangle$. Since $\langle x^r \rangle \cap \langle y^r \rangle \subset \langle x \rangle \cap \langle y \rangle = \langle x^d \rangle = \langle y^d \rangle$, it follows that

$$
\begin{aligned}
\langle x^r \rangle \cap \langle y^r \rangle &= \langle x^r \rangle \cap \langle x^d \rangle \cap \langle y^r \rangle \cap \langle y^d \rangle \\
&= \langle x^{rd/\gcd(r,d)} \rangle \cap \langle y^{rd/\gcd(r,d)} \rangle \\
&= \langle x^{rd/\gcd(r,d)} \rangle = \langle y^{rd/\gcd(r,d)} \rangle.
\end{aligned}
\tag{5.6}
$$

The equalities of line (5.6) hold because both $\langle x^{rd/\gcd(r,d)} \rangle$ and $\langle y^{rd/\gcd(r,d)} \rangle$ are the unique subgroup of $\langle x^d \rangle = \langle y^d \rangle$ of index $r/\gcd(r,d)$. Since $V' \cap V'^g = \langle x^{rd/\gcd(r,d)} \rangle$ and $V' = \langle x^r \rangle$, we conclude that $D_{V'}(g) = d/\gcd(r,d)$.

For (1), let X be a GBS tree for G. Then $V < G_x$ for some vertex $x \in X$ and $[G_x : V] < \infty$. From (2) we can say that $D_{G_x}(g) = D_{G_x}(g^k)$ implies $D_V(g) = D_V(g^k)$, and thus it suffices to establish (1) when V is a vertex stabilizer.

Let $V = G_x$ for some vertex $x \in X$. Let $X_g \subset X$ be the axis of g and $y \in X_g$ the vertex closest to x. Define the segments $\tau = [x, y]$, $\sigma_1 = [y, gy]$, $\sigma_i = g^{i-1}\sigma_1$ (for $1 < i \leqslant k$) and $\sigma = \sigma_1 \cdots \sigma_k = [y, g^k y]$. Then $[x, gx] = \tau \cdot \sigma_1 \cdot g\overline{\tau}$ and $[x, g^k x] = \tau \cdot \sigma \cdot g^k \overline{\tau}$.

Let $d = i(\sigma_1)$. Since $q(g) = \pm 1$ we have $i(\overline{\sigma}_1) = d$ as well, and so $i(\sigma_i) = i(\overline{\sigma}_i) = d$ for all i. By Lemma 3.8 it follows that $i(\sigma) = i(\overline{\sigma}) = d$. Now both $[x, gx]$ and $[x, g^k x]$ are expressed as concatenations of three segments, with matching indices along each. Hence (by Lemma 3.8) $i([x, gx]) = i([x, g^k x])$. Since $D_V(g) = i([x, gx])$ and $D_V(g^k) = [x, g^k x])$, these V-depths are equal. $\qquad\square$

Proof of Theorem 5.1. For (1), it is immediate that $\mathscr{D}(G', V) \subset \mathscr{D}(G, V)$. For the reverse inclusion, given $D_V(g) \in \mathscr{D}(G, V)$, choose k such that $g^k \in G'$. Then $D_V(g) \in \mathscr{D}(G', V)$ by Lemma 5.5(1). Conclusion (2) follows directly from Lemma 5.5(2). $\qquad\square$

Definition 5.7. Recall from (3.13) that if $\sigma = [x, gx]$ then $|q(g)| = i(\sigma)/i(\overline{\sigma})$. Thus, for any segment σ, we will call σ *unimodular* if $i(\sigma) = i(\overline{\sigma})$.

The next result gives a useful description of the depth profile.

Proposition 5.8. *Suppose $V = G_v$ for some vertex v. Define the set*

$$\mathscr{I}(v) = \{i(\sigma) \mid \sigma \text{ is a non-trivial unimodular segment with endpoints in } Gv\}.$$

Then

$$\mathscr{D}(G, V) \subseteq \mathscr{I}(v) \subseteq \mathscr{D}(G, V) \cup \{1\}.$$

In particular, if $1 \in \mathscr{D}(G, V)$ then $\mathscr{D}(G, V) = \mathscr{I}(v)$. An example where these two sets differ is $BS(1, n)$ with its standard tree as in Example 5.3. The depth profile is empty but there exist unimodular segments of index 1, namely, all segments $[x, gx]$ with g elliptic.

Proof. The first inclusion is immediate from the definitions. If g is hyperbolic with $q(g) = \pm 1$ then $D_V(g) = [V : V \cap V^g] = i([v, gv])$ and $[v, gv]$ is non-trivial and unimodular. For the second inclusion, suppose σ is a non-trivial unimodular segment with endpoints in Gv. Applying a translation in G (which does not change $i(\sigma)$), we may assume that $\partial_0(\sigma) = v$. If $i(\sigma) > 1$ then by Lemma 3.11 we have that $\sigma = [v, hv]$ with h hyperbolic, and so $i(\sigma) = D_V(h) \in \mathscr{D}(G, V)$. Otherwise, $i(\sigma) = 1$. $\square$

Remark 5.9. The definition of the depth profile has some similarities with the *scale function* of a totally disconnected locally compact group, defined by Willis [Wil94]. For the scale to be defined in our setting one must form the completion of the GBS group G with respect to the elliptic subgroups to obtain a totally disconnected locally compact group $\overline{G}$. In the case of $G = BS(m, n)$ this was carried out by Elder and Willis in [EW18]. They also gave a complete computation of the scale function. The scale of an element of $\overline{G}$ turns out to depend only on its t–exponent. Write $m = ab$, $n = bc$ where $b = \gcd(m, n)$ and let $\varepsilon \colon \overline{G} \to \mathbb{Z}$ be the t–exponent function, defined by continuous extension from G. Then the scale of g is either $|a|^{\varepsilon(g)}$ or $|c|^{-\varepsilon(g)}$, whichever one has non-negative exponent.

Recall that the depth profile uses only the hyperbolic elements of modulus ± 1, which in this case are those with t–exponent zero (and scale 1). Consider the standard HNN structure for G and let V be the index–b subgroup of the vertex group. (This choice gives the cleanest result.) Let us assume that $|m|, |n| > 1$, so that hyperbolic elements of modulus ± 1 exist. A lengthy computation using Proposition 5.8 and Lemma 3.8/Remark 3.10 shows that

$$\mathscr{D}(G, V) = \{ |a|^i |c|^j \mid i, j \in \mathbb{N} \cup \{0\}, \ i + j > 0 \}.$$

It appears that the depth profile and the scale function are detecting quite different information.

6. THE MAIN EXAMPLES

Our main examples will be lattices in $\mathrm{Aut}(X_{k,kn})$ for $k, n > 1$. We fix some notation: define a, b, c such that $b = \gcd(k, n)$, $k = ab$, and $n = bc$.

6.1. **The lattice G_1.** This group is the index 2 subgroup of $BS(k, kn)$ generated by a, tat^{-1}, and t^2. It has a labeled graph description as shown below, with two vertices and

two edges.
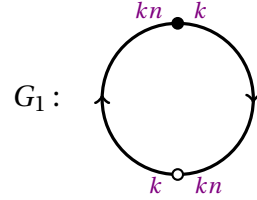
Depth profiles of $BS(k, kn)$ were computed in Example 5.3 (see also Proposition 6.2 below). Hence, for a suitable choice of elliptic subgroup $V_1 < G_1$, the depth profile is

$$\mathscr{D}(G_1, V_1) = \{n^i \mid i \in \mathbb{N} \cup \{0\}\}.$$

6.2. **The lattice** G_2**.** This group is defined by the directed labeled graph (B, μ) below. It is bipartite with two vertices v (white) and u (black), and $k + 1$ directed edges. The edges $e_1, \dots, e_k$ are directed from u to v and the edge e_0 is directed from v to u. We have $\mu(e_0) = k$, $\mu(\overline{e}_0) = kn$ and $\mu(e_i) = 1$, $\mu(\overline{e}_i) = n$ for $i \neq 0$.
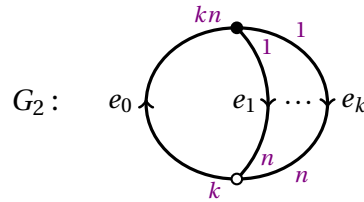
Both G_1 and G_2 are lattices in $\text{Aut}(X_{k,kn})$ by Proposition 4.5.

6.3. **The subgroup** H_2**.** We will define a finite index subgroup $H_2 < G_2$ by constructing an admissible branched covering (A, λ) of (B, μ). This subgroup will aid us in computing a depth profile for G_2.

The graph A has one vertex v_1 above v and b vertices $u_1, \dots, u_b$ above u. For each i there are k directed edges from u_i to v_1, mapping to $e_1, \dots, e_k$ respectively, and a directed edges from v_1 to u_i, mapping to e_0. If e is a directed edge above e_0 then its labels are $\lambda(e) = 1$ and $\lambda(\overline{e}) = b^2 c$. If e is a directed edge above e_i ($i \geqslant 1$) then its labels are $\lambda(e) = 1$ and $\lambda(\overline{e}) = c$.

Finally, the degree function of the branched covering is given by $d(v_1) = k$, $d(u_i) = a$, $d(e) = 1$ for every edge e above e_0, and $d(e) = a$ for every edge e above e_i ($i \geqslant 1$). See Figure 3. One may verify that the conditions of Definition 3.18 are met.
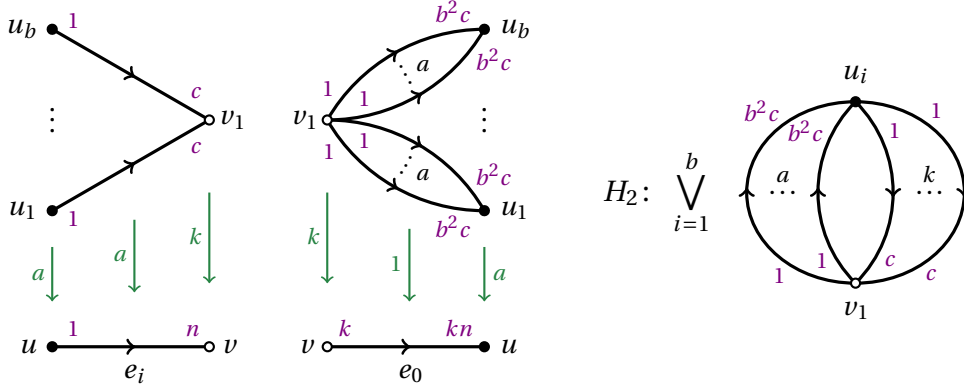
FIGURE 3. The admissible branched covering defining H_2. Above each e_i ($i \geqslant 1$) there are b edges as shown. Above e_0 there are $k = ab$ edges as shown. Overall, (A, λ) is a wedge product of b copies of a graph, joined at the vertex v_1.

By collapsing each of the edges above e_k, and then performing $k - 1$ slide moves, we find that

$$H_2 \cong \bigvee_k BS(1, n^2) \vee \bigvee_{b(k-1)} BS(c, c)$$

$$\cong BS(1, n^2) \vee \bigvee_{k-1} BS(1, 1) \vee \bigvee_{b(k-1)} BS(c, c). \tag{6.1}$$

A depth profile of H_2 can now be computed using the next result.

Proposition 6.2. *Suppose* $G = BS(1, N) \vee \bigvee_{i=1}^{r} BS(n_i, n_i)$ *for some* $r \geqslant 1$ *and suppose that* $N > 1$, *each* n_i *divides* N, *and the set* $\{n_1, \ldots, n_r, N\}$ *is closed under taking* lcm *and contains* 1. *Let* V *be the vertex group. Then*

$$\mathcal{D}(G, V) = \{N^i n_j \mid i \in \mathbb{N} \cup \{0\}, j = 1, \ldots, r\}. \tag{6.3}$$

Proof. Let X be the Bass–Serre tree for the given GBS structure of G. The subgroup V is the stabilizer of a vertex v. Note that G acts transitively on $V(X)$. Also, since $n_i = 1$ for some i, the stable letter from $BS(n_i, n_i)$ is a hyperbolic element with modulus 1 and V-depth 1. So $\mathcal{D}(G, V) = \mathcal{I}(v)$, by Proposition 5.8. That is, $\mathcal{D}(G, V)$ is the set of indices $i(\sigma)$ of non-trivial unimodular segments σ in X.

Call an edge $e \in E(X)$ *ascending* if $i(e) = 1$ and $i(\overline{e}) = N$, and *descending* if $\overline{e}$ is ascending. Note that every edge in X is either ascending, descending, or unimodular. Now every unimodular segment σ of length > 1 has one of the following forms:

(1) $\sigma_1 \sigma_2$ with σ_1, σ_2 unimodular

(2) $e_1 \tau e_2$ with τ unimodular and e_1 ascending, e_2 descending

(3) $e_1 \tau e_2$ with τ unimodular and e_1 descending, e_2 ascending.

Let D denote the right hand side of (6.3). It is easily verified that D is closed under taking lcm. We can now show that every unimodular σ has index in D by induction on length. Unimodular edges have index n_j for some j, which are in D. If σ is of type (1) then by Remark 3.10 we have $i(\sigma) = \text{lcm}(i(\sigma_1), i(\sigma_2)) \in D$. If σ is of type (2) with $i(\tau) = N^i n_j$, using Remark 3.10 one finds that $i(\sigma) = N^{i-1} n_j$ if $i \geq 1$ and $i(\sigma) = 1$ if $i = 0$. Hence $i(\sigma) \in D$. If σ is of type (3) then $i(\sigma) = N i(\tau) \in D$.

Finally, consider a segment $\sigma = e_1 \cdots e_i \tau e_1' \cdots e_i'$ where each e_k is descending, τ is a unimodular edge with index n_j, and each e_k' is ascending. Then $i(\sigma) = N^i n_j$. $\qquad\square$

Theorem 6.4. *If* $\gcd(k, n) \neq 1$ *then the lattices* $G_1, G_2 < \text{Aut}(X_{k,kn})$ *are not abstractly commensurable.*

Proof. Consider the GBS structure (6.1) for H_2 and let V_2 be its vertex group. By Theorem 5.1 and Proposition 6.2 we have

$$\mathscr{D}(G_2, V_2) = \mathscr{D}(H_2, V_2) = \{n^{2i} \mid i \in \mathbb{N} \cup \{0\}\} \cup \{n^{2i} c \mid i \in \mathbb{N} \cup \{0\}\}.$$

We also have

$$\mathscr{D}(G_1, V_1) = \{n^i \mid i \in \mathbb{N} \cup \{0\}\}$$

as mentioned earlier. Enumerating the elements of $\mathscr{D}(G_i, V_i)$ in order, notice that each element divides the next one. Taking the ratios of successive elements one obtains the sequences $(n, n, n, \dots)$ for $i = 1$ and $(c, n^2/c, c, n^2/c, \dots)$ for $i = 2$. The tails of these ratio sequences are unchanged when passing from $\mathscr{D}(G_i, V_i)$ to $\mathscr{D}(G_i, V_i)/r$ for any $r \in \mathbb{N}$, because the values of $\gcd(r, n^i)$, $\gcd(r, n^{2i})$, and $\gcd(r, n^{2i} c)$ stabilize as $i \to \infty$, all to the same number. (This number is r with all of its prime factors not dividing n removed.) Since $c \neq n$, the two tails will never agree, and so the two depth profiles are inequivalent. $\qquad\square$

Remark 6.5. If $\gcd(k, n) = 1$ then $c = n$ and the depth profiles of G_1 and G_2 coincide. However, the depth profile is not failing us as an invariant, as it turns out the groups are commensurable in this case. The labeled graph on the right hand side of Figure 3 is an admissible branched covering of the labeled graph for G_1, since $b = 1$ and $k = a$.

Example 6.6. Figure 4 illustrates G_1 and G_2 in the simplest case, when $X_{k,kn} = X_{2,4}$. In this case the lattices are commensurable to $BS(2, 4)$ and $BS(4, 16)$ respectively, which are incommensurable by [CRKZ21]. The vertical maps are admissible branched coverings and the horizontal arrows are elementary deformations. A similar phenomenon occurs

whenever $n = k > 1$: in $\mathrm{Aut}(X_{k,k^2})$ there are incommensurable lattices commensurable to $BS(k, k^2)$ and $BS(k^2, k^4)$ respectively.
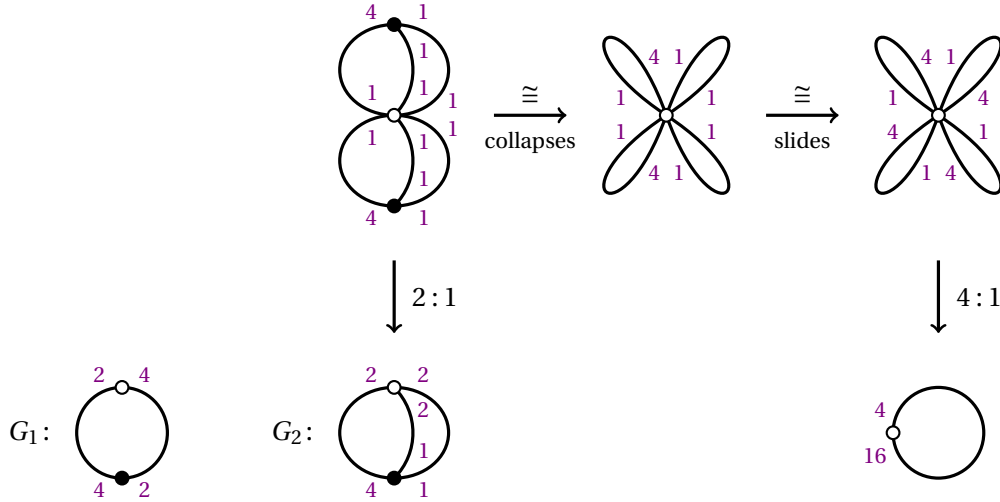

FIGURE 4. Lattices G_1, G_2 in $\mathrm{Aut}(X_{2,4})$. G_1 is an index 2 subgroup of $BS(2,4)$. G_2 contains an index 2 subgroup isomorphic to an index 4 subgroup of $BS(4,16)$ ($\not\sim BS(2,4)$). For appropriate choices of elliptic subgroups $V_i < G_i$ the depth profiles are $\mathscr{D}(G_1, V_1) = \{2^i\}$ and $\mathscr{D}(G_2, V_2) = \{4^i\}$.

Remark 6.7. Regarding Figure 4, notice that the finite-index subgroup of G_2 is a lattice both in $\mathrm{Aut}(X_{2,4})$ and $\mathrm{Aut}(X_{4,16})$, even though their "standard" lattices $BS(2,4)$ and $BS(4,16)$ are not commensurable.

7. FURTHER CASES OF $\mathrm{Aut}(X_{k,kn})$

We return to the situation of Remark 6.5, when $\gcd(k, n) = 1$.

7.1. **The lattice** G_3**.** Suppose that p is a non-trivial divisor of n (not necessarily prime) such that $p < k$. Let $l = k - p$. The labeled graph below defines a lattice $G_3 < \mathrm{Aut}(X_{k,kn})$ by Proposition 4.5. It has vertices v (white) and u (black), directed edges $e_1, \ldots, e_k$ from u to v, directed edges $f_1, \ldots, f_l$ from v to u, and a directed edge f_0 from v to u. The labels are given by $\lambda(e_i) = 1$, $\lambda(\overline{e}_i) = n$, $\lambda(f_0) = p$, $\lambda(\overline{f}_0) = pn$, and $\lambda(f_i) = 1$, $\lambda(\overline{f}_i) = n$ ($i > 0$).
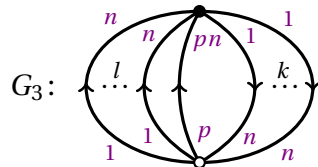
Theorem 7.1. *Suppose n has a non-trivial divisor $p \neq n$ such that $p < k$. Then the lattices $G_1, G_3 < \mathrm{Aut}(X_{k,kn})$ are not abstractly commensurable.*

Proof. Collapsing f_1 and performing $k - 1$ slide moves, we obtain

$$G_3 \cong \bigvee_k BS(1, n^2) \vee \bigvee_{k-p-1} BS(n, n) \vee BS(pn, pn)$$
$$\cong BS(1, n^2) \vee \bigvee_{k-1} BS(1, 1) \vee \bigvee_{k-p-1} BS(n, n) \vee BS(pn, pn).$$

Let V_3 be the vertex group. Proposition 6.2 provides the depth profile:

$$\mathscr{D}(G_3, V_3) = \begin{cases} \{n^{2i}\} \cup \{n^{2i+1}\} \cup \{n^{2i+1}p\} & \text{if } p < k-1 \\ \{n^{2i}\} \cup \{n^{2i+1}p\} & \text{if } p = k-1. \end{cases}$$

In both cases, enumerating the elements in order, each element divides the next. Hence there is a sequence of successive ratios that can be compared to the sequence $(n, n, n, \dots)$ arising from $\mathscr{D}(G_1, V_1)$. Exactly as in the proof of Theorem 6.4, the tails of the sequences, modulo shifting, are invariants of the equivalence classes of depth profiles.

The sequences of ratios are

the 3–periodic sequence $n, p, n/p, \dots$ if $p < k-1$
the 2–periodic sequence $np, n/p, \dots$ if $p = k-1$.

Neither of these sequences eventually agree with $(n, n, n, \dots)$, so the two depth profiles are inequivalent in both cases. □

7.2. **The lattice G_4.** Suppose $n < k$ and $k \equiv 1 \mod n$. Let $l = (k-1)/n$. The labeled graph below defines a lattice $G_4 < \mathrm{Aut}(X_{k,kn})$. It is bipartite with vertices v (white) and u (black), has directed edges $e_1, \dots, e_k$ from u to v with labels $\lambda(e_i) = 1$, $\lambda(\overline{e}_i) = n$, and directed edges $f_0, \dots, f_l$ from v to u with labels $\lambda(f_0) = 1$, $\lambda(\overline{f}_0) = n$, $\lambda(f_i) = n$, $\lambda(\overline{f}_i) = n^2$ ($i \geq 1$).
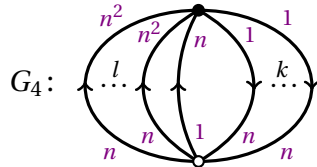
Theorem 7.2. *Suppose $n < k$ and $k \equiv 1 \mod n$. Then the lattices $G_1, G_4 < \mathrm{Aut}(X_{k,kn})$ are not abstractly commensurable.*

Proof. Collapsing f_0 and then performing $2l + k - 1$ slide moves yields

$$G_4 \cong \bigvee_l BS(n^2, n^2) \vee \bigvee_k BS(1, n^2)$$
$$\cong BS(1, n^2) \vee \bigvee_{l+k-1} BS(1, 1).$$

The depth profile is $\{ n^{2i} \mid i \in \mathbb{N} \cup \{0\} \}$, which is inequivalent to the depth profile of G_1. $\square$

7.3. **The remaining cases.** There are two remaining cases of $X_{k,kn}$ not covered by Theorems 6.4, 7.1, and 7.2. The first is when every prime factor of n is greater than k. The second is when n is prime, $n < k$, and $k \not\equiv 0, 1 \mod n$ (for instance, $X_{5,15}$). The constructions seen thus far all result in lattices having equivalent depth profiles. Ad hoc arguments seem to indicate that some of these examples (in the second case) are still incommensurable, but a complete presentable proof is elusive so far.

8. CAYLEY GRAPHS

Given a finitely generated group G let S be a symmetric finite generating set that does not contain 1. The *Cayley graph* $\mathrm{Cay}(G, S)$ is a connected graph with edges labeled by elements of S, defined as follows. The vertex set is G. For every $g \in G$ and $s \in S$ there is a unique edge e with $\partial_0(e) = g$ and $\partial_1(e) = gs$. This edge is assigned the label s. Thus, $\bar{e}$ has label s^{-1}.

We say that G_1 and G_2 *admit isomorphic Cayley graphs* if there exist generating sets S_1, S_2 as above such that $\mathrm{Cay}(G_1, S_1)$ and $\mathrm{Cay}(G_2, S_2)$ are isomorphic as unlabeled graphs. Note that this is not a transitive relation (see [HS10]).

There are several interesting examples of groups that are quite different from one another admitting isomorphic Cayley graphs. Most involve torsion in an essential way, since finite groups of the same cardinality always admit isomorphic Cayley graphs. If A and B are two such finite groups, then $A \wr \mathbb{Z}$ and $B \wr \mathbb{Z}$ admit isomorphic Cayley graphs, by [Dyu00]. (Such groups are never finitely presented, however.) Similarly, if G_A and G_B are extensions of A and B respectively by a group Q, then G_A and G_B admit isomorphic Cayley graphs (see [MMV16, proof of Corollary 1.13] for instance).

If one seeks incommensurable *torsion-free* groups with isomorphic Cayley graphs, then some well known examples can be found among lattices in products of locally finite trees. This phenomenon was first observed and discussed by Wise in [Wis96]; the lattices of Burger and Mozes [BM00] also provide examples. In [DK23] Dergacheva and Klyachko constructed a pair of incommensurable torsion-free groups with isomorphic Cayley graphs, via amalgams of Baumslag–Solitar groups. Our lattices here provide new

examples, which moreover are also coherent. It is an open question whether there exist coherent lattices in products of trees [Wis07, Problem 10.10].

If a group acts cocompactly on a connected CW complex X, freely and transitively on the vertices, then the 1–skeleton $X^{(1)}$ is a Cayley graph for G. This is the situation for the torsion-free examples just mentioned. Unfortunately, our lattices G_i don't act in this way on $X_{k,kn}$. Instead we have the following result.

Proposition 8.1. *Suppose G_1 and G_2 act cocompactly on a connected graph Γ, freely on the vertices, with a common vertex orbit. Then G_1 and G_2 admit isomorphic Cayley graphs.*

Having a vertex orbit in common is important. For instance, two groups may act on the same graph, with the same number of vertex orbits, and that is not enough. In the group $G = \mathbb{Z} \times \mathbb{Z}/2$ with any Cayley graph Γ, the subgroups $\mathbb{Z} \times \{0\}$ and $2\mathbb{Z} \times \mathbb{Z}/2$ both act on Γ with two vertex orbits, but they do not admit isomorphic Cayley graphs, by [HS10] (see also [Löh13]).

Proof. The assumptions imply that Γ is locally finite with bounded valence. Also, there is a vertex $v_0 \in V(\Gamma)$ such that $G_1 v_0 = G_2 v_0$. Let V_0 denote this vertex orbit. Because the action of G_1 is cocompact, there is a number C such that every vertex of Γ has distance at most C from V_0.

Now let P be the set of paths in Γ of the form $\alpha \cdot e \cdot \beta$ where

- $e \in E(\Gamma)$
- α is a shortest path in Γ from V_0 to $\partial_0 e$
- β is a shortest path in Γ from $\partial_1 e$ to V_0.

We build a new graph Δ' as follows: $V(\Delta')$ is the set V_0 and $E(\Delta')$ is P. That is, each $p \in P$ has initial and terminal endpoints $\partial_0 p$ and $\partial_1 p$ in V_0, which defines an edge in Δ'. (Note that P is closed under the involution $p \mapsto \overline{p}$.) Finally, define Δ from Δ' by eliminating all loops and duplicate edges, if any exist.

We claim that Δ is connected. Suppose $\gamma = (e_1, \ldots, e_n)$ is any path in Γ with endpoints in V_0. For each i let α_i be a shortest path from V_0 to $\partial_0 e_i$. Then $(e_1 \cdot \overline{\alpha}_2) \cdot (\alpha_2 \cdot e_2 \cdot \overline{\alpha}_3) \cdots (\alpha_n \cdot e_n)$ is a concatenation of paths in P from $\partial_0 \gamma$ to $\partial_1 \gamma$.

It is immediate that both G_1 and G_2 act on Δ, freely and transitively on the vertices. Next let

$$S_i = \{ g \in G_i \mid d_\Delta(v_0, g v_0) = 1 \}$$

for $i = 1, 2$. These sets are finite because $d_\Delta(v_0, g v_0) = 1$ implies $d_\Gamma(v_0, g v_0) \leq 2C + 1$ and Γ is locally finite. Finally, $\mathrm{Cay}(G_i, S_i) \cong \Delta$ since every path in P has a unique translate

under the action of G_i with initial vertex v_0. That is, every edge of Δ has a unique G_i–translate which is an edge from v_0 to sv_0 for some $s \in S_i$. □

Corollary 8.2. *The lattices $G_1, G_2, G_3, G_4 < \mathrm{Aut}(X_{k,kn})$ admit isomorphic Cayley graphs.*

Proof. Recall that the labeled graphs defining G_i have two vertices, black and white. Lifting this coloring to the common Bass–Serre tree $T_{k,kn}$ we get a bipartite vertex coloring. Now lift this vertex coloring to the vertices of $X_{k,kn}$ using π. Each branching line has only vertices of one color, and every strip has opposite vertex colors on its two sides. Now note that the white vertices and the black vertices are exactly the vertex orbits under any of the group actions. Hence Proposition 8.1 applies, with Γ the 1–skeleton of $X_{k,kn}$. □

9. QUESTIONS

The main question is the following:

Question 9.1. For which pairs (m, n) does $\mathrm{Aut}(X_{m,n})$ contain incommensurable lattices?

The main open cases are the two cases of $\mathrm{Aut}(X_{k,kn})$ from Section 7.3 and the case when $\gcd(m, n) \neq 1$, $m \nmid n$, and $n \nmid m$.

Question 9.2. What are the uniform lattices in $\mathrm{Aut}(X_{m,n})$ with torsion? Are there uniform lattices that are not virtually torsion-free?

For any (A, λ) there is a straightforward construction of a lattice in $\mathrm{Aut}(X_{(A,\lambda)})$ with 2–torsion, via a graph of infinite dihedral groups. These examples contain torsion-free subgroups of index 2. It would be much more interesting to find lattices with higher-order torsion.

If T is a locally finite tree then every uniform lattice in $\mathrm{Aut}(T)$ is virtually torsion-free, by [KPS73]. On the other hand, Hughes recently constructed a uniform lattice in $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ that is not virtually torsion-free [Hug22]. The reasoning used in [Hug22] seems unlikely to apply in our setting, as it depends on the existence of a simple uniform lattice.

Question 9.3. Does $\mathrm{Aut}(X_{m,n})$ contain non-uniform lattices?

Carbone [Car01] has shown that $\mathrm{Aut}(T)$ contains non-uniform lattices, for any uniform locally finite tree T. Rémy [Rém99] has shown that certain Kac-Moody groups over finite fields are irreducible non-uniform lattices in $\mathrm{Aut}(T \times T)$.

Note that $\mathrm{Aut}(X_{m,m}) \cong \mathrm{Aut}(T_{2m}) \times D_\infty$ for $m > 1$, so it contains reducible non-uniform lattices, by [Car01]. Here T_{2m} is the *undirected* regular tree of valence $2m$. Hence we are really asking about *irreducible* non-uniform lattices if $m = n$.

Question 9.4. When do $\text{Aut}(X_{m,n})$ and $\text{Aut}(X_{m',n'})$ contain isomorphic lattices?

We have seen that $\text{Aut}(X_{k,k^2})$ and $\text{Aut}(X_{k^2,k^4})$ contain isomorphic lattices (namely, the finite-index subgroup from Figure 4), even though their "standard" lattices $BS(k, k^2)$ and $BS(k^2, k^4)$ are not commensurable. This failure of rigidity for lattices in $\text{Aut}(X_{m,n})$ is intriguing.

Finally, all of the above can be studied for the more general groups $\text{Aut}(X_{(A,\lambda)})$.

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