Final Exam
Algebraic Topology
May 10, 2007

You may apply theorems from the course, but please give the name or statement of the theorem.

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1. Consider the long exact sequence of homotopy groups

$$\cdots \to \pi_n(A,B,x_0) \xrightarrow{i_*} \pi_n(X,B,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,B,x_0) \to \cdots$$

for a triple (X,A,B). Show that the sequence is exact at the $\pi_n(X,B,x_0)$ term.

First, $\mathrm{Im}(i_*) \subset \mathrm{Ker}(j_*)$*: recall that an element* $[f] \in \pi_n(X,A,x_0)$ *is trivial if and only if* f *is homotopic (through maps of triples) to a map with image in* A*. So* $j_*(i_*([g])) = [j \circ i \circ g]$ *is trivial for any* $[g] \in \pi_n(A,B,x_0)$ *since* $j \circ i \circ g$ *already has image in* A*.*

Next, $\mathrm{Ker}(j_*) \subset \mathrm{Im}(i_*)$*: if* $[f] \in \mathrm{Ker}(j_*)$ *then after a homotopy we can assume that* f *maps* $(I^n, \partial I^n, J^{n-1})$ *into* (A, B, x_0)*, and now it represents an element of* $i_*(\pi_n(A,B,x_0))$*.*

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2. **(a)** State the Lefschetz fixed point theorem, and define the Lefschetz number $\tau(f)$ of a map $f\colon X \to X$, where X is a (retract of a) finite simplicial complex. (Note that this includes compact CW complexes.)

(b) Give the homology groups and cohomology groups of $\mathbb{C}\mathrm{P}^n$ in $\mathbb{Z}$-coefficients, and also describe the cup product structure (no proof needed).

(a) *If* X *is a retract of a finite simplicial complex then its homology groups are finitely generated, and non-zero in only finitely many dimensions. Define the Lefschetz number* $\tau(f)$ *to be* $\sum_i (-1)^i \,\mathrm{tr}(f_*\colon H_i(X)/T_i \to H_i(X)/T_i)$*, where* $H_i(X)/T_i$ *is the homology group with torsion factored out. The Lefschetz fixed point theorem says that if* $\tau(f) \neq 0$ *then* f *has a fixed point.*

(b) $H_i(\mathbb{C}\mathrm{P}^n)$ *is* $\mathbb{Z}$ *for* i *even and* $0 \le i \le 2n$ *and* 0 *otherwise. The cohomology groups* $H^i(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ *are the same. If* α *is a generator of* $H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ *then* α^i *generates* $H^{2i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ *for* $i \le n$*. That is,* $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ *with* $|\alpha| = 2$*.*

(c) If $f^*\colon H^i(\mathbb{C}\mathrm{P}^n) \to H^i(\mathbb{C}\mathrm{P}^n)$ is multiplication by d, what is the map $f_*\colon H_i(\mathbb{C}\mathrm{P}^n) \to H_i(\mathbb{C}\mathrm{P}^n)$? Explain why, using the universal coefficient theorem.

(d) Prove that every map $f\colon \mathbb{C}\mathrm{P}^{2k} \to \mathbb{C}\mathrm{P}^{2k}$ has a fixed point. [Hint: use the cup product.]

(c) *For every* i *the group* $H_{i-1}(\mathbb{C}\mathrm{P}^n)$ *is free, so* $\mathrm{Ext}(H_{i-1}(\mathbb{C}\mathrm{P}^n),\mathbb{Z}) = 0$*, and hence* $H^i(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathrm{Hom}(H_i(\mathbb{C}\mathrm{P}^n),\mathbb{Z})$ *by the universal coefficient theorem. Moreover, by naturality we have the commutative diagram*

$$\begin{array}{ccc}
H^i(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) & \xrightarrow[\cong]{h} & \mathrm{Hom}(H_i(\mathbb{C}\mathrm{P}^n),\mathbb{Z}) \\
\uparrow{\scriptstyle \cdot d} & & \uparrow{\scriptstyle (f_*)^*} \\
H^i(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) & \xrightarrow[\cong]{h} & \mathrm{Hom}(H_i(\mathbb{C}\mathrm{P}^n),\mathbb{Z})
\end{array}$$

and therefore $(f_*)^*$ *is nultiplication by* d*. This implies that* f_* *is also multiplication by* d*.*

(d) *Let* $n = 2k$. *Let* $\alpha \in H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ *be a generator and suppose* $f^*(\alpha) = d\alpha$. *Then* $f^*(\alpha^i) = (f^*(\alpha))^i = d^i\alpha^i$, *so* $f^*\colon H^{2i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to H^{2i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ *is multiplication by* d^i. *By part (c) the map* $f_*\colon H_{2i}(\mathbb{C}\mathrm{P}^n) \to H_{2i}(\mathbb{C}\mathrm{P}^n)$ *is also multiplication by* d^i. *So* $\tau(f) = \sum_{i=0}^n (-1)^{2i} d^i = \sum_{i=0}^n d^i$. *Now* $d^0 = 1$ *and* $\sum_{i=1}^n d^i$ *is even, so* $\tau(f) \neq 0$.

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3. Prove the *extension lemma*: Let (X, A) be a finite-dimensional CW pair and Y a path connected space such that $\pi_{n-1}(Y) = 0$ for all n for which $X - A$ has an n-cell. Then every map $f\colon A \to Y$ can be extended to a map $X \to Y$. [Hint: define the extension cell-by-cell, in increasing dimensions.]

First, extend to the 0-cells of $X - A$ *by sending them to any points of* Y. *Next, assume that* f *has been extended to all of* $X^{(n-1)} \cup A$. *Let* e *be an* n*-cell of* $X - A$ *with attaching map* $\phi\colon \partial D^n \to X^{(n-1)}$. *Then* $[f \circ \phi] \in \pi_{n-1}(Y)$ *and this group is trivial by assumption. So* $f \circ \phi$ *extends to a map* $D^n \to Y$, *and this extension joins with* f *to give an extension to* $(X^{(n-1)} \cup e) \cup A$. *The other* n*-cells can be handled simultaneously, since their interiors are disjoint. By induction on* n, f *extends to all of* X.

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4. Let (X, A) be a CW pair with A contractible. Use the extension lemma to show that A is a retract of X.

Consider the identity map $i\colon A \to A$, *and note that* $\pi_{n-1}(A) = 0$ *for all* n. *Hence, by the extension lemma,* i *extends to a map* $r\colon X \to A$. *(We must either assume* X *finite-dimensional, or that the extension lemma holds without this assumption (it does).) Now,* r *is a retraction since it is the identity on* A.

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5. Let $K \subset S^3$ be a *knot*, i.e. an embedded circle. Let N be a closed ε-neighborhood of K which is homeomorphic to the solid torus $D^2 \times S^1$. Let $X = S^3 - \mathrm{int}(N)$. Note that N and X are both compact 3-manifolds with boundary, with common boundary $X \cap N$, which is a torus.

Use the Mayer-Vietoris sequence to compute the first homology group of X. [It turns out the answer does not depend on whether K is actually knotted or not!]

We have the following portion of the Mayer-Vietoris sequence for S^3 *expressed as the union of* N *and* X:

$$\to H_2(S^3) \to H_1(N \cap X) \to H_1(N) \oplus H_1(X) \to H_1(S^3) \to$$

and note that $H_2(S^3) = H_1(S^3) = 0$. *Hence* $H_1(N \cap X) \cong H_1(N) \oplus H_1(X)$. *Next,* $H_1(N \cap X) \cong H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ *and* $H_1(N) \cong \mathbb{Z}$ *since* N *deformation retracts onto a circle. Then we have* $H_1(X) \cong H_1(N \cap X)/H_1(N) \cong (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ *for some* n. *Since* $H_1(N) \oplus H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, *we must have* $n = 1$ *and* $H_1(X) \cong \mathbb{Z}$.

For $H_2(X)$ *we can proceed similarly, but the Mayer-Vietoris sequence gives slightly less information. Namely, the sequence*

$$\to H_3(N) \oplus H_3(X) \to H_3(S^3) \to H_2(N \cap X) \to H_2(N) \oplus H_2(X) \to H_2(S^3) \to$$

yields the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to H_2(X) \to 0$$

since $H_2(N) = H_2(S^3) = 0$. *(The groups* $H_3(N)$ *and* $H_3(X)$ *vanish since* N *and* X *are 3-manifolds with non-empty boundary.) Thus* $H_2(X)$ *is finite cyclic.*

In fact, $H_2(X) = 0$*, as we can see using Lefschetz duality. First,* $H_3(X, \partial X) \cong \mathbb{Z}$*, and in the long exact sequence for the pair* $(X, \partial X)$*, the boundary map takes a fundamental class to a fundamental class of the boundary. Thus, the exact sequence*

$$\to H_3(X, \partial X) \to H_2(\partial X) \to H_2(X) \to H_2(X, \partial X)$$

has the form

$$\to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} H_2(X) \to H_2(X, \partial X).$$

Hence $H_2(X)$ *is a subgroup of* $H_2(X, \partial X)$*, and must be zero if* $H_2(X, \partial X)$ *has no torsion. Indeed,* $H_2(X, \partial X) \cong H^1(X)$ *by Lefschetz duality, and this latter group is* $\mathrm{Hom}(H_1(X), \mathbb{Z}) \cong \mathbb{Z}$*.*

REMARK: *If* K *is unknotted then* X *is homeomorphic to the solid torus* $D^2 \times S^1$*. The calculations above show that* X *has the same homology as* $D^2 \times S^1$*, even if* K *is knotted. So homology does not detect knottedness.*