Exam I Solutions
Topology (Math 5863)

1(a) If X and Y are topological spaces, let $\mathcal{C}(X,Y)$ be the set of continuous maps from X to Y. Give the definition of the *compact-open* topology on $\mathcal{C}(X,Y)$ by giving a subbasis.

(b) Now suppose that X is locally compact and Hausdorff, and give $\mathcal{C}(X,Y)$ the compact-open topology. Define the *evaluation map* $e\colon X \times \mathcal{C}(X,Y) \to Y$ by the formula $e(x,f) = f(x)$. Prove that this map is continuous.

[Hint: recall that X locally compact implies that for every open neighborhood W of x, there is an open neighborhood U with $\overline{U} \subset W$ and $\overline{U}$ compact.]

Solution.

(a) For each compact set $C \subset X$ and open set $U \subset Y$ let

$$S(C,U) = \{f \in \mathcal{C}(X,Y) \mid f(C) \subset U\}.$$

The set of all such sets $S(C,U)$ is a subbasis for the compact-open topology on $\mathcal{C}(X,Y)$.

(b) Let (x,f) be a point in $X \times \mathcal{C}(X,Y)$ and $V \subset Y$ a neighborhood of $e(x,f)$. We need to find a neighborhood of (x,f) that is mapped by e into V.

Since f is continuous and $f(x) \in V$, the set $W = f^{-1}(V)$ is an open neighborhood of x in X. By local compactness and the Hausdorff property, there is a neighborhood U of x such that $\overline{U} \subset W$ and $\overline{U}$ is compact. Now $U \times S(\overline{U}, V)$ is an open neighborhood of (x,f) since $f(\overline{U}) \subset V$. Also, any $(x', f') \in U \times S(\overline{U}, V)$ is mapped to V by e, since $x' \in U$ and f' takes U into V.

2(a) Define what it means for $r\colon X \to A$ to be a *retraction*, where A is a subspace of X.

(b) Let $i\colon A \to X$ be inclusion and let $r\colon X \to A$ be a retraction, and pick a basepoint $a_0 \in A$. Show that the induced homomorphism $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.

(c) Show that there is no retraction of the "solid torus" $S^1 \times D^2$ to the boundary torus $S^1 \times S^1$.

Solution.

(a) r is a retraction if r is continuous and $r(a) = a$ for all $a \in A$.

(b) The retraction property means that $r \circ i$ is the identity map on A. Hence, $r_* \circ i_* = (r \circ i)_* = (\mathrm{id})_* = \mathrm{id}$, the identity homomorphism on $\pi_1(A, a_0)$. In particular, this composition is a bijection, and it follows that i_* is injective and r_* is surjective.

(c) We know that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong 1$, and therefore $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^1 \times D^2) \cong \mathbb{Z} \times 1 \cong \mathbb{Z}$. By part (b), if there is a retraction then there will be a corresponding injective homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. However, this is impossible. For instance, if $(1,0)$ and $(0,1)$ map to m and n respectively, then $(n,0)$ and $(0,m)$ both map to mn.

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3. Let $p\colon E \to B$ be a covering map. Choose $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$.

(a) Define the *lifting correspondence* $\Phi\colon \pi_1(B, b_0) \to p^{-1}(b_0)$.

(b) Show that Φ is surjective, and that Φ is injective if E is simply connected. State carefully any results that you use.

SOLUTION. Note, we should also assume that E is path connected.

(A) For each $[f] \in \pi_1(B, b_0)$ let $\tilde{f}$ be the unique lift of f starting at e_0. Then $\Phi([f]) = \tilde{f}(1)$. This is well defined because if $[f] = [g]$ then any path homotopy from f to g lifts to a path homotopy from $\tilde{f}$ to $\tilde{g}$, showing that $\tilde{f}$ and $\tilde{g}$ have the same endpoints in $p^{-1}(b_0)$.

(B) For surjectivity, suppose $e_1 \in p^{-1}(b_0)$. Since E is path connected, there is a path $\tilde{f}$ from e_0 to e_1. Then $\tilde{f}$ is a lift of the loop $f = p \circ \tilde{f}$, and it starts at e_0, and therefore $e_1 = \tilde{f}(1) = \Phi([f])$.

For injectivity, suppose $\Phi([f]) = \Phi([g]) = e_1 \in p^{-1}(b_0)$. The lifts $\tilde{f}$ and $\tilde{g}$ (of f and g respectively, starting at e_0) are paths in E from e_0 to e_1. Since E is simply connected, there is a path homotopy F from $\tilde{f}$ to $\tilde{g}$. Then $p \circ F$ is a path homotopy from f to g, and therefore $[f] = [g]$.

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4. Let $h\colon I \to X$ be a path from x_0 to x_1.

(a) Give the definition of the change-of-basepoint homomorphism $\beta_h\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$. [Or, in Munkres notation, the homomorphism $\widehat{h}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$.]

(b) Prove that β_h (or $\widehat{h}$) is a homomorphism, and an isomorphism.

SOLUTION. (the For Munkres version see Theorem 52.1.)

(A) We define $\beta_h([f]) = [h \cdot f \cdot \overline{h}]$. This is well defined because $[h \cdot f \cdot \overline{h}] = [h][f][\overline{h}]$ and multiplication of path homotopy classes is well defined.

(B) First, $\beta_h([f])\beta_h([g]) = [h \cdot f \cdot \overline{h}][h \cdot g \cdot \overline{h}] = [h][f][\overline{h}][h][g][\overline{h}] = [h][f][g][\overline{h}] = [h \cdot f \cdot g \cdot \overline{h}] = \beta_h([f \cdot g]) = \beta_h([f][g])$, and so β_h is a homomorphism. Next we claim that $\beta_{\overline{h}}$ and β_h are inverse homomorphisms (and therefore are isomorphims). We verify: $\beta_h(\beta_{\overline{h}}([f])) = [h \cdot \overline{h} \cdot f \cdot h \cdot \overline{h}] = [f]$ and $\beta_{\overline{h}}(\beta_h([f])) = [\overline{h} \cdot h \cdot f \cdot \overline{h} \cdot h] = [f]$.

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5. Let $p\colon E \to B$ be a covering map with B connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$ then $p^{-1}(b)$ has k elements for every $b \in B$.

SOLUTION.

Define sets $A \subset B$ and $C \subset B$ as follows: $A = \{b \in B \mid |p^{-1}(b)| = k\}$ and $C = \{b \in B \mid |p^{-1}(b)| \neq k\}$. Clearly $A \cap C = \emptyset$ and $A \cup C = B$. We claim that A and B are both open sets.

If $b \in A$ then there is an evenly covered neighborhood U of b. Then $p^{-1}(U) \cong V_1 \cup \cdots \cup V_k$, with each slice V_i mapping by p homeomorphically onto U. There are k slices, because $p^{-1}(b)$ has k elements, and it contains one element of each slice. The same is true of any $b' \in U$, and therefore $U \subset A$.

If $b \in C$ then an evenly covered neighborhood U of b has preimage equal to the union of a collection of slices, having the same cardinality as $p^{-1}(b)$. Every fiber $p^{-1}(b')$ has this same cardinality, which is not k. Hence, $U \subset C$.

Thus, A and C are open sets. Since A contains b_0 and B is connected, it must be the case that $C = \emptyset$, and $B = A$.
