Homework solutions, 3/2/14

Munkres §58 #2.

(a) $B^2 \times S^1$. This deformation retracts to the circle $\{0\} \times S^1$, with fundamental group $\mathbb{Z}$.

(b) $T - \{p\}$. This deformation retracts onto a subspace homeomorphic to $S^1 \vee S^1$. Consider the unit square with sides identified to form T, and remove p from the center. Then push out to the boundary of the square, which is really $S^1 \vee S^1$ when the identifications are made.

(c) $S^1 \times I$. This deformation retracts to the circle $S^1 \times \{0\}$.

(d) $S^1 \times \mathbb{R}$. This deformation retracts to the circle $S^1 \times \{0\}$.

(e) $R^3 - Z$ where Z is the union of the non-negative x, y, and z axes. Removing the origin only, the space deformation retracts onto the sphere S^2. Removing the three rays, it deformation retracts onto the sphere minus three points. Finally, the sphere minus three points deformation retracts onto a copy of $S^1 \vee S^1$. (The first puncture gives an open disk, then there are two more punctures.)

(f) $\{x : ||x|| > 1\}$ deformation retracts onto S^1 (of radius larger than 1).

(g) $\{x : ||x|| \geq 1\}$ deformation retracts onto S^1.

(h) $\{x : ||x|| < 1\}$ is contractible.

(i) $S^1 \cup (\mathbb{R}_+ \times 0)$ deformation retracts onto S^1.

(j) $S^1 \cup (\mathbb{R}_+ \times \mathbb{R})$ deformation retracts onto S^1.

(k) $S^1 \cup (\mathbb{R} \times 0)$ is homotopy equivalent to $S^1 \vee S^1$.

(l) $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$ is contractible (it is star-convex).

Munkres §58 #3.

If f is a homotopy equivalence from X to Y, then there is a homotopy inverse g from Y to X, and the symmetry in the definition immediately implies that g is also a homotopy equivalence. So the relation is symmetric. It is reflexive since the identity map on X is a homotopy equivalence (with itself as the homotopy inverse). For transitivity, suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are homotopy equivalences with homotopy inverses $f'\colon Y \to X$ and $g'\colon Z \to Y$. We claim that $f' \circ g'$ is a homotopy inverse to $g \circ f$, and therefore $g \circ f$ is a homotopy equivalence from X to Z. We have:

$$(f' \circ g') \circ (g \circ f) = f' \circ (g' \circ g) \circ f \simeq f' \circ (\mathrm{id}_Y) \circ f = f' \circ f \simeq \mathrm{id}_X$$

and

$$(g \circ f) \circ (f' \circ g') = g \circ (f \circ f') \circ g' \simeq g \circ (\mathrm{id}_Y) \circ g' = g \circ g' \simeq \mathrm{id}_Z\,.$$

Munkres §58 #9.

(a) The problem would be relatively easy if not for basepoints. I recommend drawing a picture with all of the paths discussed below. We have a continuous map $h\colon S^1 \to S^1$, taking x_0 to x_1. Let g be a generator of $\pi_1(S^1, b_0)$. The recipe for computing the *degree* of h is: take any path α from b_0 to x_0 and any path β from b_0 to x_1; then the degree is the unique integer d such that $h_*(\widehat{\alpha}(g)) = (\widehat{\beta}(g))^d$. [Having chosen the generator of $\pi_1(S^1, b_0)$, the elements $\widehat{\alpha}(g)$ and $\widehat{\beta}(g)$ are generators of $\pi_1(S^1, x_0)$

and $\pi_1(S^1, x_1)$ respectively. So, the homomorphism $h_*\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by d. We need to know the generators first to distinguish between multiplication by d and by $-d$.]

Now consider a new basepoint x_0' and let $x_1' = h(x_0')$. Let δ be a path from x_0 to x_0'. Then $\alpha \cdot \delta$ is a path from b_0 to x_0' and $\beta \cdot (h \circ \delta)$ is a path from b_0 to x_1'. Recall the following properties of change-of-basepoint isomorphisms:

$$\widehat{(\alpha \cdot \delta)} \;=\; \widehat{\delta} \circ \widehat{\alpha} \tag{1}$$

and

$$(\beta \cdot \widehat{(h \circ \delta)}) \;=\; \widehat{(h \circ \delta)} \circ \widehat{\beta}. \tag{2}$$

Another property of change-of-basepoint isomorphisms we will need is compatibility with induced homomorphisms:

$$\begin{array}{ccc} \pi_1(S^1, x_0) & \xrightarrow{h_*} & \pi_1(S^1, x_1) \\ \Big\downarrow \widehat{\delta} & & \Big\downarrow \widehat{h \circ \delta} \\ \pi_1(S^1, x_0') & \xrightarrow{h_*} & \pi_1(S^1, x_1') \end{array} \tag{3}$$

Let d be the degree of h using x_0 and x_1. Thus we have

$$h_*(\widehat{\alpha}(g)) = (\widehat{\beta}(g))^d = \widehat{\beta}(g^d). \tag{4}$$

We want to show that $h_*(\widehat{(\alpha \cdot \delta)}(g)) = (\beta \cdot \widehat{(h \circ \delta)})(g^d)$, since this equation is a valid recipe for finding the degree using the basepoints x_0' and x_1'. Rewriting using (1) and (2), we want to show that $h_*(\widehat{\delta}(\widehat{\alpha}(g))) = \widehat{(h \circ \delta)}(\widehat{\beta}(g^d))$. The left side equals $\widehat{(h \circ \delta)}(h_*(\widehat{\alpha}(g)))$ by (3), which equals $\widehat{(h \circ \delta)}(\widehat{\beta}(g^d))$ by (4), and we are done.

Remark. According to the recipe, we are free to choose *any* paths to get the change-of-basepoint isomorphisms. Our choice of $h \circ \delta$ (after picking δ) allowed us to use (3), which greatly simplified the proof. If a choice is up to you, always try to make it as compatibly as possible with the situation.

(b) By (a) we can assume that $x_0 = b_0$ and use g as the generator of $\pi_1(S^1, b_0)$. Let $x_1 = h(b_0)$ and $x_2 = k(b_0)$. Let α be a path from b_0 to x_1, so that

$$h_*(g) = \widehat{\alpha}(g)^{d_h}, \qquad d_h = \text{degree}(h).$$

By Lemma 58.4 there is a path δ from x_1 to x_2 such that $k_* = \widehat{\delta} \circ h_*$. The path $\beta = \alpha \cdot \delta$ is a path from b_0 to x_2, and so the degree d_k of k satisfies

$$k_*(g) = \widehat{\beta}(g)^{d_k} = \widehat{\beta}(g^{d_k}) = \widehat{\delta}(\widehat{\alpha}(g^{d_k})).$$

Using the Lemma 58.4 property we get $\widehat{\delta}(h_*(g)) = \widehat{\delta}(\widehat{\alpha}(g^{d_k}))$, and since $\widehat{\delta}$ is a isomorphism, we can apply the inverse to obtain $h_*(g) = \widehat{\alpha}(g^{d_k}) = \widehat{\alpha}(g)^{d_k}$. This shows that $d_h = d_k$.

(c) Briefly: multiplication by d_k followed by multiplication by d_h is multiplication by $d_k d_h$. More rigorously:

Assume $x_0 = b_0$ and use g as the generator fo $\pi_1(S^1, b_0)$. Let $x_1 = k(b_0)$ and $x_2 = h(x_1)$. Let α be a path from b_0 to x_1, and β a path from b_0 to x_2. We have

$$k_*(g) = \widehat{\alpha}(g)^{d_k} \quad \text{and} \quad h_*(\widehat{\alpha}(g)) = \widehat{\beta}(g)^{d_h}$$

where d_k and d_h are the degrees of k and h respectively. Now

$$(h \circ k)_*(g) = h_*(k_*(g)) = h_*(\widehat{\alpha}(g)^{d_k}) = (h_*(\widehat{\alpha}(g))^{d_k} = (\widehat{\beta}(g)^{d_h})^{d_k} = (\widehat{\beta}(g))^{d_h d_k},$$

and so $h \circ k$ has degree $d_h d_k$.

(d) Use the generator $g = [\omega_1]$ for $\pi_1(S^1, b_0)$. If h is a constant map, then h is homotopic to the constant map $e_{b_0} = \omega_0$ (which has the same degree by (b)). Then $h_*(g) = [\omega_0] = g^0$ and the degree is 0.

If h is the identity map then $h_*(g) = g = g^1$ and the degree is 1.

If h is the reflection $(\cos\theta, \sin\theta) \mapsto (\cos\theta, -\sin\theta)$ then this is the same as $e^{i\theta} \mapsto e^{i(-\theta)}$, or $z \mapsto z^{-1}$. Thus $h \circ \omega_1 = \omega_{-1}$ and $h_*(g) = g^{-1}$. The degree is -1.

If h is the map $z \mapsto z^n$ then $h \circ \omega_1 = \omega_n$ and so $h_*(g) = g^n$. The degree is n.

(e) We show that h is homotopic to a map of the form $z \mapsto z^n$; the degree is then n. Two homotopic maps of the same degree n will then be homotopic to each other, since they are homotopic to the same map $z \mapsto z^n$.

First, we can modify h by a homotopy (using rotations in S^1) to arrange that $h(b_0) = b_0$. The degree does not change, by (b).

Next consider the domain S^1 as the quotient space of I with its endpoints identified, with b_0 the image of the endpoints. Let $q\colon I \to S^1$ be this quotient map. The map $f = h \circ q\colon I \to S^1$ is a loop in S^1 at b_0. (Informally, this is just h considered as a *loop*.) Let $\tilde{f}$ be the lift of f to the covering space $p\colon \mathbb{R} \to S^1$, starting at 0. Let $n = \tilde{f}(1)$. There is a path homotopy $\tilde{f}_t\colon I \to \mathbb{R}$ with $\tilde{f}_0 = \tilde{f}$ and $\tilde{f}_1 = \widetilde{\omega}_n$. Then $p \circ \tilde{f}_t$ is a path homotopy in S^1 from f to ω_n.

The path homotopy $f_t = p \circ \tilde{f}_t\colon I \to S^1$ is constant on the fibers of q (the only non-trivial fibers are the endpoints of I, and these map to b_0 for all t). Hence f_t factors as $g_t \circ q$; that is, the homotopy $g_t\colon S^1 \to S^1$ is defined, and it is a homotopy from $g_0 = h$ to g_1, which is the map $z \mapsto z^n$ (because q followed by $z \mapsto z^n$ is precisely ω_n). Hence every map h is homotopic to $z \mapsto z^n$ for some n.

Hatcher §1.1 #11.
i_* is surjective: given $[f] \in \pi_1(X, x_0)$, note that since I is path connected, its image under f lies within a single path component of X. Its image contains x_0, so this path component is X_0 and $f(I) \subset X_0$. Define $g\colon I \to X_0$ by $g(t) = f(t)$, and note that $i \circ g = f$, and so $i_*([g]) = [f]$.

i_* is injective: suppose $i_*([f]) = i_*([g])$. Then $[i \circ f] = [i \circ g]$ and so there is a path homotopy in X from $i \circ f$ to $i \circ g$. This is a map $F\colon I \times I \to X$, and since $I \times I$ is path connected, the image lies entirely within X_0. Thus there is a path homotopy $G\colon I \times I \to X_0$ defined by $G(s,t) = F(s,t)$, which is a path homotopy in X_0 from f to g. Hence $[f] = [g]$ in $\pi_1(X_0, x_0)$.

Hatcher §1.1 #13. **Note:** "homotopic" in Hatcher means "path homotopic"
($\Rightarrow$): suppose $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ is onto. Let $f\colon I \to X$ be a path in X with endpoints in A. Let $a_0\colon I \to X$ and $a_1\colon I \to X$ be paths in A from x_0 to $f(0)$ and from $f(1)$ to x_0 respectively. Then $a_0 \cdot f \cdot a_1$ is a loop in X at x_0. By assumption $i_*([g]) = [a_0 \cdot f \cdot a_1])$ for some loop g in A. That is, there is a path homotopy F from $a_0 \cdot f \cdot a_1$ to $i \circ g$, a loop in X with image contained in A. Restricting this path homotopy to a subset $I' \times I$ of $I \times I$, we obtain a homotopy from f to a

path with image in A. Turning this homotopy into a *path* homotopy is quite tricky to explain – I'll have to wait until next class.
($\Leftarrow$): suppose every path in X with endpoints in A is homotopic to a path in A. Given $[f] \in \pi_1(X, x_0)$ we have that f is (path) homotopic to a path g in A. This is a loop at x_0 in A so $[g] \in \pi_1(A, x_0)$. Also $i \circ g = f$, so $i_*([g]) = [f]$ and i_* is surjective.
