Final Exam Solutions
Topology II (Math 5863)

1(a) Give the definition of a homotopy equivalence.

(b) If $f\colon X \to Y$ is a homotopy equivalence, prove that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism. List carefully any theorems or properties that you use.

(c) Give an example of two spaces that are homotopy equivalent but not homeomorphic. (Explain clearly why they are not homeomorphic.)

Solution.

(a) It is a continuous map $f\colon X \to Y$ such that there exists another continuous map $g\colon Y \to X$ with $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

(b) Let f and g be as above. Then $f_* \circ g_* = (f \circ g)_* = (\mathrm{id}_Y)_* = \mathrm{id}$ and similarly $g_* \circ f_* = \mathrm{id}$. Therefore f_* and g_* are inverse isomorphisms of groups. Properties used:

- if $F \simeq G$ then $F_* = G_*$
- $(\mathrm{id}_Z)_*$ is the identity homomorphism of $\pi_1(Z, z_0)$
- $(F \circ G)_* = F_* \circ G_*$

(c) One example would be a point and any contractible space, such as $\mathbb{R}$. They are homotopy equivalent because $\mathbb{R}$ deformation retracts onto a point, and deformation retractions are homotopy equivalences. They are not homeomorphic because there is not even a bijection between them (they do not have the same number of points).

2(a) Recall that the Möbius band X is the quotient space of the square $[0,1] \times [0,1]$ obtained by identifying $(0,t)$ with $(1, 1-t)$ for all $t \in [0,1]$. What is the fundamental group of the Möbius band? Explain.

(b) The boundary of the Möbius band is a single loop. Which element of the fundamental group does it represent?

(c) If you glue two copies of the Möbius band together along their boundary curves, you get a surface. Use van Kampen's theorem to find a presentation for the fundamental group of this surface.

Solution.

(a) The Möbius band deformation retracts onto its core circle, which is the subspace $[0,1] \times \{\frac{1}{2}\}$ with endpoints identified. Thus its fundamental group is infinite cyclic, generated by the homotopy class of the loop $[0,1] \times \{\frac{1}{2}\}$. Call this class a, so the group is $\langle a \rangle$.

(b) The boundary curve is the loop made up of the two paths $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ (joined end to end). Under the deformation retraction to $[0,1] \times \{\frac{1}{2}\}$, it wraps twice around $[0,1] \times \{\frac{1}{2}\}$, and so it represents $a^2 \in \langle a \rangle$.

(C) Join two copies of the Möbius band as described to form the space Z. Let A and B be open neighborhoods of the two Möbius bands which deformation retract onto the Möbius bands. The intersection of A and B is a neighborhood of the common boundary curve C which deformation retracts onto C.

The fundamental groups of A, B, and C are each infinite cyclic. Pick a basepoint $x_0 \in C$ and write $\pi_1(A, x_0) = \langle a \rangle$, $\pi_1(B, x_0) = \langle b \rangle$ and $\pi_1(C, x_0) = \langle c \rangle$. Let $(i_A)_*\colon \pi_1(C, x_0) \to \pi_1(A, x_0)$ and $(i_B)_*\colon \pi_1(C, x_0) \to \pi_1(B, x_0)$ be the homomorphisms induced by the inclusion maps $C \to A$ and $C \to B$. By part (b) we know that $(i_A)_*(c) = a^2$ and $(i_B)_*(c) = b^2$.

Van Kampen's theorem says that $\pi_1(Z, x_0) \cong \dfrac{\pi_1(A, x_0) * \pi_1(B, x_0)}{N}$ where N is normally generated by the element $(i_A)_*(c)(i_B)_*(c)^{-1}$. Thus $\pi_1(Z, x_0)$ has presentation $\langle\, a, b \mid a^2 b^{-2} = 1 \,\rangle$, or equivalently, $\langle\, a, b \mid a^2 = b^2 \,\rangle$.

Incidentally, Z is the Klein bottle and this is another presentation of its fundamental group.

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3. Give an outline of the argument showing that there is an isomorphism from the fundamental group of the circle to $\mathbb{Z}$. You do not need to prove the various lemmas that you use, but state them clearly.

SOLUTION. Let $p\colon \mathbb{R} \to S^1$ be the usual covering map and $s_0 \in S^1$ the usual basepoint (with $p^{-1}(s_0)$ equal to the integers in $\mathbb{R}$). The isomorphism is given by the lifting correspondence $\Phi\colon \pi_1(S^1, s_0) \to p^{-1}(s_0)$, defined as follows. Given $[f] \in \pi_1(S^1, s_0)$ there is a unique lift $\tilde{f}$ to $\mathbb{R}$ starting at 0. Then, $\Phi([f])$ is defined to be the endpoint $\tilde{f}(1)$. This is well defined because, if $[f] = [g]$, then any path homotopy from f to g lifts to a path homotopy from $\tilde{f}$ to $\tilde{g}$; in particular, $\tilde{f}$ and $\tilde{g}$ have the same endpoints and $\tilde{f}(1) = \tilde{g}(1)$.

Φ is a homomorphism: given $[f]$ and $[g]$ suppose $\Phi([f]) = m$ and $\Phi([g]) = n$. Let $\tilde{f}$ and $\tilde{g}$ be the lifts of f and g to $\mathbb{R}$ starting at 0. Then $\tilde{f}(1) = m$ and $\tilde{g}(1) = n$. Let $\tau_m\colon \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x + m$. Then, $\tau_m \circ \tilde{g}$ is the lift of g starting at m, and so $\tilde{f} \cdot (\tau_m \circ \tilde{g})$ is the lift of $f \cdot g$ starting at 0. Hence, $\Phi([f][g])$ is the endpoint of this lift, which is $(\tau_m \circ \tilde{g})(1) = \tilde{g}(1) + m = n + m$.

Φ is surjective: for any $n \in \mathbb{Z}$, let $\tilde{f}$ be a path in $\mathbb{R}$ from 0 to n, and let $f = p \circ \tilde{f}$. Then, by construction, $\Phi([f]) = n$.

Φ is injective: if $\Phi([f]) = 0$ then $\tilde{f}$ is a loop in $\mathbb{R}$. Let $\tilde{f}_t$ be a path homotopy (in $\mathbb{R}$) from this loop to the constant path at 0. Then, $p \circ \tilde{f}_t$ is a path homotopy from f to the constant path at s_0. Thus, $[f] = 0$ in $\pi_1(S^1, s_0)$.

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4. Recall that $\mathbb{R}P^2$ is a surface that has a 2–sheeted covering map $S^2 \to \mathbb{R}P^2$.

(a) What is the fundamental group of $\mathbb{R}P^2$?

(b) Describe the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

(c) What is the fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$? Can you describe (or list) all of its elements explicitly?

SOLUTION.

(A) The fundamental group is $\mathbb{Z}/2\mathbb{Z}$. (Since S^2 is simply connected, the fundemantal group must be a group of order 2.)

(B) The universal cover is made of infinitely many copies of S^2 joined end-to-end, like a string of beads. The points where two adjacent spheres touch map to the basepoint of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and each sphere maps to one of the copies of $\mathbb{R}P^2$ by the covering map $S^2 \to \mathbb{R}P^2$. The spheres map to the copies of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ in an alternating fashion.

(C) By van Kampen's theorem, the fundamental group is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, or $\langle\, a, b \mid a^2 = 1, b^2 = 1\,\rangle$. Its elements are represented by reduced words in the letters a and b. Since $a = a^{-1}$ and $b = b^{-1}$, the letter a can only be followed by the letter b and vice versa. Thus the reduced words are exactly the alternating sequences of a's and b's.

5(a) A path connected, locally path connected, semi-locally simply connected space X has fundamental group $\mathbb{Z}/13\mathbb{Z}$. How many (equivalence classes of) path connected covering spaces does X have? Why?

(b) Same question, but now X has fundamental group $\mathbb{Z}/6\mathbb{Z}$. Also, how many of its covering spaces are regular?

SOLUTION.

(A) The equivalence classes of path connected covering spaces correspond bijectively with the conjugacy classes of subgroups of $\mathbb{Z}/13\mathbb{Z}$. Since this is an abelian group, two subgroups are conjugate if and only if they are equal. Thus, the equivalence classes of covering spaces correspond to the subgroups. The only subgroups of this group are the group itself and the trivial subgroup (because 13 is prime). Hence, the only path connected covering spaces of X are itself and its universal cover.

(B) All the same comments apply to this case, except that now there are more than two subgroups of $\mathbb{Z}/6\mathbb{Z}$. Writing the group as $\langle a \mid a^6 = 1\rangle$, the subgroups are $\{1\}$, $\{1, a^3\}$, $\{1, a^2, a^4\}$, and $\{1, a, a^2, a^3, a^4, a^5\}$. So there are four covering spaces. They are all regular because these subgroups are all normal subgroups. (Every subgroup of an abelian group is normal.)

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6. Describe carefully how to build a space whose fundamental group is given by the presentation $\langle a, b \mid ababab = 1,\ a^5 = 1\rangle$.

SOLUTION. We will build a cell complex X with the given fundamental group. Start with one 0-cell and add two 1-cells, labeled a and b. This space so far is the one skeleton $X^{(1)}$ and it is homeomorphic to the figure eight. Next, we will attach two 2-cells via attaching maps $\phi_1, \phi_2 \colon\ S^1 \to X^{(1)}$.

There maps can be considered as loops in $X^{(1)}$ and we simply have ϕ_1 be the loop which traverses the path labeled *ababab*. The map ϕ_2 will be the loop which traverses the path labeled *aaaaa*.

A corollary of Van Kampen's theorem says that the fundamental group of X will be given by the presentation $\langle a, b, \mid ababab = 1, a^5 = 1\rangle$.

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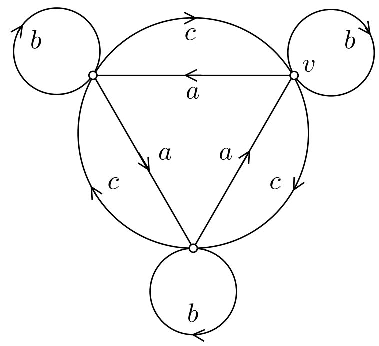
7. Consider the covering space $p\colon X \to S^1 \vee S^1 \vee S^1$ shown in the picture below.



Find a free generating set for the subgroup $H = p_*(\pi_1(X,v))$ of $\langle a,b,c\rangle$ (the fundamental group of $\pi_1(S^1 \vee S^1 \vee S^1, x_0)$). Also, say as much as you can about the properties of H.

Solution. First we find a collection of loops in X corresponding to a free basis for the fundamental group $\pi_1(X,v)$. Choose a maximal subtree $T \subset X$, for instance the union of the two edges labeled by a that are incident to v. There will be a free generator for each edge not in T. Thus, there are 7 free generators for this group. As based loops in X, they are given by the following edge-paths: b, $a^{-1}ba$, aba^{-1}, a^3, ca, $a^{-1}ca^{-1}$, ac.

The subgroup H is generated by the elements of $\langle a,b,c\rangle$ which are the images of the homotopy classes of the loops described above. The labeling scheme is set up so that the loop labeled by ac maps to a loop representing the word ac, and so on. Thus, $H = \langle b, a^{-1}ba, aba^{-1}, a^3, ca, a^{-1}ca^{-1}, ac\rangle$.

We can also say that H is a subgroup of $\langle a,b,c\rangle$ of index 3, because X is a 3–sheeted covering space of $S^1 \vee S^1 \vee S^1$. Furthermore, there are covering transformations of X consisting of rotations by $2\pi/3$ in the picture above. These self-homeomorphisms preserve the labeling, and so they commute with the projection map to $S^1 \vee S^1 \vee S^1$. These transformations can take v to any other vertex, so this is a regular covering. Therefore, H is a normal subgroup.

8(a) State the *lifting criterion* for a map $f\colon (X,x_0) \to (B,b_0)$, where $p\colon (E,e_0) \to (B,b_0)$ is a covering map and all spaces are path connected and locally path connected.

(b) If X is path connected, what is the uniqueness property for lifts of maps $f\colon X \to B$?

(c) Define the notion of an *equivalence* between covering spaces $p\colon E \to B$ and $p'\colon E' \to B$.

(d) Define the notion of a *covering transformation* for a covering space $p\colon E \to B$.

(e) If E, B are path connected, show that if h, k are covering transformations that agree at a point $e \in E$, then $h = k$.

Solution.

(a) The lifting criterion states that a lift $\tilde{f}\colon (X,x_0) \to (E,e_0)$ exists if and only if $f_*(\pi_1(X,x_0)) \subset p_*(\pi_1(E,e_0)$. Note that $\tilde{f}$ being a lift means that $f = p \circ \tilde{f}$ (and $\tilde{f}(x_0) = e_0$).

(b) The uniqueness property is that two lifts are equal if and only if they agree at one point.

(c) An equivalence is a homeomorphism $h\colon E \to E'$ such that $p' \circ h = p$.

(d) A covering transformation is an equivalence from a covering space to itself.

(E) Think of $e \in E$ as a basepoint and let $b = p(e)$ and $e' = h(e)$. We are given covering transformations h, k which both take e to e'. Note that a covering transformation taking e to e' is exactly a lift of the continuous map $p\colon (E, e) \to (B, b)$ to the covering space $p\colon (E, e') \to (B, b)$. By part (b), $h = k$.
