Final Exam
Topology II (Math 5863)
May 8, 2014

Choose six problems. If you do more, please say which six you want graded.

1(a) Give the definition of a homotopy equivalence.

(b) If $f\colon X \to Y$ is a homotopy equivalence, prove that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism. List carefully any theorems or properties that you use.

(c) Give an example of two spaces that are homotopy equivalent but not homeomorphic. (Explain clearly why they are not homeomorphic.)

2(a) Recall that the Möbius band X is the quotient space of the square $[0, 1] \times [0, 1]$ obtained by identifying $(0, t)$ with $(1, 1 - t)$ for all $t \in [0, 1]$. What is the fundamental group of the Möbius band? Explain.

(b) The boundary of the Möbius band is a single loop. Which element of the fundamental group does it represent?

(c) If you glue two copies of the Möbius band together along their boundary curves, you get a surface. Use van Kampen's theorem to find a presentation for the fundamental group of this surface.

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3. Give an outline of the argument showing that there is an isomorphism from the fundamental group of the circle to $\mathbb{Z}$. You do not need to prove the various lemmas that you use, but state them clearly.

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4. Recall that $\mathbb{R}P^2$ is a surface that has a 2–sheeted covering map $S^2 \to \mathbb{R}P^2$.

(a) What is the fundamental group of $\mathbb{R}P^2$?

(b) Describe the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

(c) What is the fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$? Can you describe (or list) all of its elements explicitly?

5(a) A path connected, locally path connected, semi-locally simply connected space X has fundamental group $\mathbb{Z}/13\mathbb{Z}$. How many (equivalence classes of) path connected covering spaces does X have? Why?

(b) Same question, but now X has fundamental group $\mathbb{Z}/6\mathbb{Z}$. Also, how many of its covering spaces are regular?

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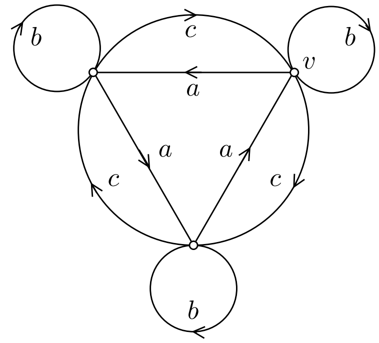
6. Describe carefully how to build a space whose fundamental group is given by the presentation $\langle a, b \mid ababab = 1,\ a^5 = 1\rangle$.

(PLEASE TURN OVER →)

7(a) Consider the covering space $p\colon X \to S^1 \vee S^1 \vee S^1$ shown in the picture below.



Find a free generating set for the subgroup $H = p_*(\pi_1(X, v))$ of $\langle a, b, c\rangle$ (the fundamental group of $\pi_1(S^1 \vee S^1 \vee S^1, x_0)$). Also, say as much as you can about the properties of H.

8(a) State the *lifting criterion* for a map $f\colon (X, x_0) \to (B, b_0)$, where $p\colon (E, e_0) \to (B, b_0)$ is a covering map and all spaces are path connected and locally path connected.

(b) If X is path connected, what is the uniqueness property for lifts of maps $f\colon X \to B$?

(c) Define the notion of an *equivalence* between covering spaces $p\colon E \to B$ and $p'\colon E' \to B$.

(d) Define the notion of a *covering transformation* for a covering space $p\colon E \to B$.

(e) If E, B are path connected, show that if h, k are covering transformations that agree at a point $e \in E$, then $h = k$.