Exam I
Topology II (Math 5863)
March 6, 2014

Choose four problems. If you do five, please say which four you want graded.

1(a) If X and Y are topological spaces, let $\mathcal{C}(X,Y)$ be the set of continuous maps from X to Y. Give the definition of the *compact-open* topology on $\mathcal{C}(X,Y)$ by giving a subbasis.

(b) Now suppose that X is locally compact and Hausdorff, and give $\mathcal{C}(X,Y)$ the compact-open topology. Define the *evaluation map* $e\colon X \times \mathcal{C}(X,Y) \to Y$ by the formula $e(x,f) = f(x)$. Prove that this map is continuous.

[Hint: recall that X locally compact implies that for every open neighborhood W of x, there is an open neighborhood U with $\overline{U} \subset W$ and $\overline{U}$ compact.]

2(a) Define what it means for $r\colon X \to A$ to be a *retraction*, where A is a subspace of X.

(b) Let $i\colon A \to X$ be inclusion and let $r\colon X \to A$ be a retraction, and pick a basepoint $a_0 \in A$. Show that the induced homomorphism $i_*\colon \pi_1(A,a_0) \to \pi_1(X,a_0)$ is injective.

(c) Show that there is no retraction of the "solid torus" $S^1 \times D^2$ to the boundary torus $S^1 \times S^1$.

3. Let $p\colon E \to B$ be a covering map. Choose $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$.

(a) Define the *lifting correspondence* $\Phi\colon \pi_1(B,b_0) \to p^{-1}(b_0)$.

(b) Show that Φ is surjective, and that Φ is injective if E is simply connected. State carefully any results that you use.

4. Let $h\colon I \to X$ be a path from x_0 to x_1.

(a) Give the definition of the change-of-basepoint homomorphism $\beta_h\colon \pi_1(X,x_1) \to \pi_1(X,x_0)$. [Or, in Munkres notation, the homomorphism $\widehat{h}\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$.]

(b) Prove that β_h (or $\widehat{h}$) is a homomorphism, and an isomorphism.

5. Let $p\colon E \to B$ be a covering map with B connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$ then $p^{-1}(b)$ has k elements for every $b \in B$.