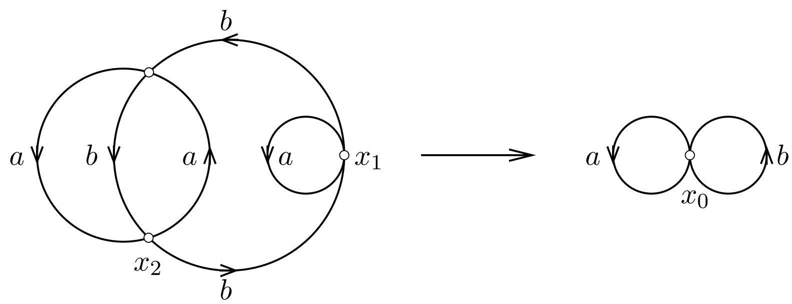
Final Exam Solutions
Topology II
May 10, 2006

1. (20 points) Below is a covering space $p\colon \widetilde{X} \to X$ of the figure eight space $X = S^1 \vee S^1$. Let $G = \pi_1(X, x_0) = \langle a, b\rangle$ and $H_1 = p_*(\pi_1(\widetilde{X}, x_1))$.



(a) Find a free generating set for H_1. What is its rank? What is its index in G?

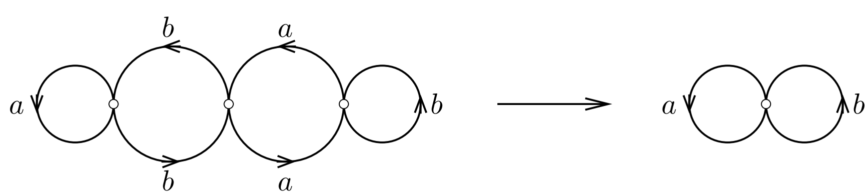
(b) Let $H_2 = p_*(\pi_1(\widetilde{X}, x_2))$. Find a generating set for H_2, using the same tree as for H_1. What is the relationship between H_1 and H_2? Be as specific as you can.

(c) Find a subgroup H_3 of G having the same index as H_1, such that H_1 and H_3 are not conjugate in G. [Hint: find a covering space of X with the right properties.]

(A) *Let $T \subset \widetilde{X}$ be the maximal tree consisting of the path from x_1 to x_2 labeled by the word b^2. Then H_1 has free generators a, b^3, b^2ab^{-1}, and bab^{-2}. It has rank 4 and since the covering has three sheets, $[G : H_1] = 3$.*

(B) *The free generators for H_2 (using T again) are $b^{-2}ab^2$, b^3, ab, and $b^{-1}a$. The subgroups are conjugate since the based coverings differ by a change of basepoint only. The path in T from x_1 to x_2 gives the conjugating element, in this case b^2. Indeed, by comparing generating sets, we see that $H_2 = b^{-2}H_1b^2$.*

(C) *Two subgroups are conjugate if and only if they are the image subgroups of two isomorphic covering spaces. Thus we need a covering space with 3 sheets that is not isomorphic to $p\colon \widetilde{X} \to X$ (for any choice of basepoint). For example, the following cover will do, since it is not even homeomorphic to $\widetilde{X}$. Here $H_3 = \langle bab^{-1}, b^2, a^2, a^{-1}ba\rangle$ (using the middle vertex as basepoint, and the top middle two edges as the maximal tree).*



2. (10 points) Let X be a space which possesses a universal cover. Let $p_1\colon X_1 \to X$ and $p_2\colon X_2 \to X_1$ be covering maps. Prove that $p_1 \circ p_2\colon X_2 \to X$ is a covering map. [Assume X is locally path connected.]

We shall use the following simple fact: if $f\colon A \to B$ and $g\colon B \to C$ are continuous maps such that $g \circ f$ is a homeomorphism and f is surjective, then f and g are homeomorphisms. To prove this, let $h = g \circ f$ and note that since h is bijective, f must be injective and g surjective. Thus f is bijective, and hence so is g (since $g = h \circ f^{-1}$). Now $f^{-1} = h^{-1} \circ g$ and $g^{-1} = f \circ h^{-1}$ are continuous, being compositions of continuous maps.

Returning to the problem, $p_1 \circ p_2$ is continuous and surjective, so we only need to show that X has an evenly covered neighborhood at every point. Given $x \in X$ let U be a path connected neighborhood of x that is evenly covered by the universal cover $p\colon \widetilde{X} \to X$. Let $\{U_\alpha\}$ be the slices of $p^{-1}(U)$, so that $p|_{U_\alpha}$ is a homeomorphism for each α.

By the lifting criterion (applied to p_1) the map $p\colon \widetilde{X} \to X$ lifts to $\tilde{p}\colon \widetilde{X} \to X_1$, and by applying the lifting criterion again to the cover p_2, we get a lift $p_0\colon \widetilde{X} \to X_2$. That is, $p_0\colon \widetilde{X} \to X_2$ has the property that $p = p_1 \circ p_2 \circ p_0$.

For each α let $V_\alpha \subset X_2$ be the set $p_0(U_\alpha)$. The fact proved above shows that $(p_1 \circ p_2)|_{V_\alpha}$ is a homeomorphism of V_α onto U. It remains to prove that the sets $\{V_\alpha\}$ partition $(p_1 \circ p_2)^{-1}(U)$.

Note that $p_1\colon X_1 \to X$ and $p_2\colon X_2 \to X_1$ have the unique lifting property for maps of connected spaces, and it follows directly that $p_2 \circ p_1$ has this property as well. Now consider the inclusion map $U \to X$ and the maps $(p_2 \circ p_1)|_{V_\alpha}{}^{-1}\colon U \to V_\alpha \hookrightarrow X_2$, which are lifts of this inclusion. Two such maps either are equal or differ at every point. Therefore, V_α and V_β are equal or are disjoint, for every α and β.

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3. (15 points) Let $p\colon \mathbb{R} \to S^1$ be the standard covering map $t \mapsto (\cos(2\pi t), \sin(2\pi t)) = e^{2\pi i t}$. Let $b = p(0) \in S^1$ be the basepoint.

(a) Define the *lifting correspondence* $\phi\colon \pi_1(S^1, b) \to \mathbb{Z}$ and show that it is well defined.

(b) Show that ϕ is injective.

(A) *Note first that $\mathbb{Z} = p^{-1}(b)$. We define $\phi([f]) = \tilde{f}(1)$ where $\tilde{f}$ is the unique lift of $f\colon I \to S^1$ with $\tilde{f}(0) = 0$. If $[f] = [g]$ then there is a path homotopy between f and g, and this homotopy lifts to a path homotopy between $\tilde{g}$ and $\tilde{f}$. Hence $\tilde{f}(1) = \tilde{g}(1)$.*

(B) *If $\phi([f]) = \phi([g])$ then $\tilde{f}$ and $\tilde{g}$ are paths in $\mathbb{R}$ with the same endpoints. Then there is a path homotopy $H\colon I \times I \to \mathbb{R}$ between $\tilde{f}$ and $\tilde{g}$ (the straight line homotopy, for example). The map $p \circ H$ is a path homotopy between f and g and so $[f] = [g]$.*

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4. (15 points) Let $A \subset \mathbb{R}^3$ be a compact subspace and let $f\colon A \to A$ be a continuous map. Show that there is a continuous map $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $g|_A = f$. State carefully any major theorems that you use.

Let $i\colon A \to \mathbb{R}^3$ be inclusion and $\pi_i\colon \mathbb{R}^3 \to \mathbb{R}$ the projection onto the ith coordinate. Let $f_i = \pi_i \circ f$. Thus $i \circ f\colon A \to \mathbb{R}^3$ is given by (f_1, f_2, f_3).

Since $\mathbb{R}^3$ is normal and A is closed, the Tietze extension theorem says that there are maps $g_i\colon \mathbb{R}^3 \to \mathbb{R}$ such that $g_i|_A = f_i$. Now $g = (g_1, g_2, g_3)$ is a map $\mathbb{R}^3 \to \mathbb{R}^3$ which restricts to f on A.

Tietze extension theorem: *If X is normal and $A \subset X$ is closed, then every continuous map $f\colon A \to \mathbb{R}$ extends to a continuous map $g\colon X \to \mathbb{R}$ (that is, $g|_A = f$). The same statement also holds with $\mathbb{R}$ replaced by any closed interval.*

5. (15 points) Let $S^3 \subset \mathbb{R}^3$ be the unit sphere. Let $X = S^3/\sim$ where $x \sim -x$ for all $x \in S^3$. Note that X is usually called $\mathbb{R}P^3$, or *real 3-dimensional projective space.*

(a) Prove that S^3 is simply connected.
(b) Explain why S^3 is a covering space of X. How many sheets does it have? What is the fundamental group of X?

(A) *We have seen two proofs of this. Here is one: let N, S be two points in S^3 and let $U = S^3 - N$, $V = S^3 - S$. By stereographic projection, U and V are homeomorphic to $\mathbb{R}^3$ and are simply connected. Also $U \cap V$ is path connected. Hence, by van Kampen's theorem, $\pi_1(S^3)$ is trivial. Also S^3 is path connected, and so it is simply connected.*

(B) *Let $q\colon S^3 \to X$ be the quotient map. Given $x \in S^3$ let U be a neighborhood of x that is disjoint from $-U$. Then $U \cup -U$ is a saturated open set, so $q(U)$ is open in X. The restriction $q|_U$ is injective, and therefore is a homeomorphism of U onto $q(U)$. The same is true of $-U$, so $q(U)$ is evenly covered by q. Also, q is clearly continuous and surjective.*

The number of sheets of the map $S^3 \to X$ is two, since each point has two preimage points. Since S^3 and X are connected, the image subgroup of S^3 has index two in the fundamental group of X. Hence the fundamental group of X has two elements, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

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6. (15 points) A covering space $p\colon \widetilde{X} \to X$ is shown below. The spaces are 2-dimensional manifolds (surfaces). Note that each embedded circle C_i maps homeomorphically onto the circle C, and each component of $\widetilde{X} - \bigcup_i C_i$ maps by a homeomorphism to $X - C$. (If you cut $\widetilde{X}$ along the curves C_i, take one of the pieces, and glue its boundary components together, you get a copy of X.)

[THE PICTURE IS FROM PAGE 73 OF HATCHER.]

(a) Draw a loop γ on X which represents a non-trivial element of $\pi_1(X, x_0)$, and which is not in the image subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. Using lifts to $\widetilde{X}$, explain why γ has this property.

(b) Describe informally the covering translations (or automorphisms) of $p\colon \widetilde{X} \to X$. How many are there?

(A) *Draw a loop on the surface that crosses C in a single point. Then any lift will be a path from C_i to C_{i+1} for some i. Since this lift is not a loop, the original curve does not represent an element of $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$.*

(B) *If $x \in X$ is any point, let $p^{-1}(x) = \{x_0, x_1, x_2, x_3, x_4\}$. Every covering translation takes x_0 to some x_i. Furthermore, if two covering translations agree at a point, they are equal. Hence there are at most five covering translations (including the identity).*

There are in fact five covering translations, given by rotations by multiples of $2\pi/5$ in the picture.