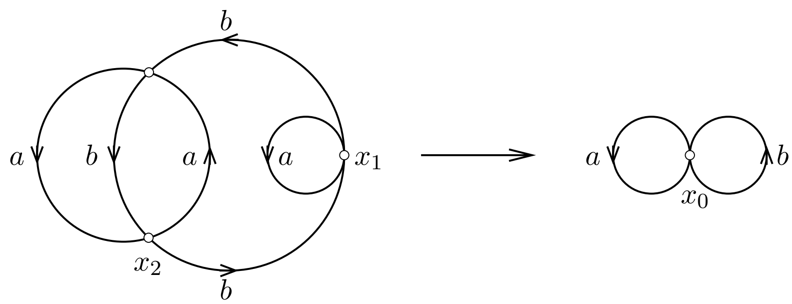
Final Exam
Topology II
May 10, 2006

1. (20 points) Below is a covering space $p\colon \widetilde{X} \to X$ of the figure eight space $X = S^1 \vee S^1$. Let $G = \pi_1(X, x_0) = \langle a, b\rangle$ and $H_1 = p_*(\pi_1(\widetilde{X}, x_1))$.



(a) Find a free generating set for H_1. What is its rank? What is its index in G?

(b) Let $H_2 = p_*(\pi_1(\widetilde{X}, x_2))$. Find a generating set for H_2, using the same tree as for H_1. What is the relationship between H_1 and H_2? Be as specific as you can.

(c) Find a subgroup H_3 of G having the same index as H_1, such that H_1 and H_3 are not conjugate in G. [Hint: find a covering space of X with the right properties.]

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2. (10 points) Let X be a space which possesses a universal cover. Let $p_1\colon X_1 \to X$ and $p_2\colon X_2 \to X_1$ be covering maps. Prove that $p_1 \circ p_2\colon X_2 \to X$ is a covering map. [Assume X is locally path connected.]

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3. (15 points) Let $p\colon \mathbb{R} \to S^1$ be the standard covering map $t \mapsto (\cos(2\pi t), \sin(2\pi t)) = e^{2\pi i t}$. Let $b = p(0) \in S^1$ be the basepoint.

(a) Define the *lifting correspondence* $\phi\colon \pi_1(S^1, b) \to \mathbb{Z}$ and show that it is well defined.

(b) Show that ϕ is injective.

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4. (15 points) Let $A \subset \mathbb{R}^3$ be a compact subspace and let $f\colon A \to A$ be a continuous map. Show that there is a continuous map $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $g|_A = f$. State carefully any major theorems that you use.

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5. (15 points) Let $S^3 \subset \mathbb{R}^3$ be the unit sphere. Let $X = S^3/\sim$ where $x \sim -x$ for all $x \in S^3$. Note that X is usually called $\mathbb{R}P^3$, or *real* 3-*dimensional projective space.*

(a) Prove that S^3 is simply connected.

(b) Explain why S^3 is a covering space of X. How many sheets does it have? What is the fundamental group of X?

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6. (15 points) A covering space $p\colon \widetilde{X} \to X$ is shown below. The spaces are 2-dimensional manifolds (surfaces). Note that each embedded circle C_i maps homeomorphically onto the circle C, and each component of $\widetilde{X} - \bigcup_i C_i$ maps by a homeomorphism to $X - C$. (If you cut $\widetilde{X}$ along the curves C_i, take one of the pieces, and glue its boundary components together, you get a copy of X.)

[The picture is from page 73 of Hatcher.]

(a) Draw a loop γ on X which represents a non-trivial element of $\pi_1(X, x_0)$, and which is not in the image subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. Using lifts to $\widetilde{X}$, explain why γ has this property.

(b) Describe informally the covering translations (or automorphisms) of $p\colon \widetilde{X} \to X$. How many are there?