Exam III Solutions
Topology II
Due May 2, 2006

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1. (a) Prove that if $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is an n-sheeted covering space then $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ is a subgroup of $\pi_1(X, x_0)$ of index n.

(b) Find two 2-sheeted covering spaces of the torus $S^1 \times S^1$ that are not isomorphic to each other (as covering spaces). Can you find three?

(A) *We must assume (as usual) that $\widetilde{X}$ is connected. Let $G = \pi_1(X, x_0)$ and $H = p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. Write $p^{-1}(x_0)$ as $\{\tilde{x}_0, \ldots, \tilde{x}_{n-1}\}$. For any $[\gamma_1], [\gamma_2] \in G$ we claim that the cosets $[\gamma_1]H$ and $[\gamma_2]H$ are equal if and only if the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ starting at $\tilde{x}_0$ have the same endpoint $\tilde{x}_i$. Also every $\tilde{x}_i$ can be joined to $\tilde{x}_0$ by a path $\tilde{\gamma}$ which represents some element $[\gamma] = [p \circ \tilde{\gamma}] \in G$. Hence the left cosets of H are in one-to-one correspondence with $p^{-1}(x_0)$, and $[G : H] = n$.*

To prove the claim, note that $[\gamma_1]H = [\gamma_2]H$ if and only if $[\gamma_1]^{-1}[\gamma_2]H = H$, i.e. if and only if $[\overline{\gamma}_1\gamma_2] \in H$. This occurs (by a theorem given in class) if and only if $\overline{\gamma}_1 * \gamma_2$ lifts to a loop in $\widetilde{X}$. Hence if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same endpoint, $[\gamma_1]H = [\gamma_2]H$. Conversely, if $\tilde{\gamma}_1(1) \neq \tilde{\gamma}_2(1)$ then $\overline{\tilde{\gamma}}_1 * \tilde{\gamma}_2$ is a lift of $\overline{\gamma}_1 * \gamma_2$ that is not a loop, and so $[\gamma_1]H \neq [\gamma_2]H$.

(B) *The first covering is $p_1\colon S^1 \times S^1 \to S^1 \times S^1$ given by $(z, w) \mapsto (z^2, w)$ in complex notation. The image subgroup in the fundamental group of $S^1 \times S^1$ is $2\mathbb{Z} \times \mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. The second covering is $p_2\colon S^1 \times S^1 \to S^1 \times S^1$ given by $(z, w) \mapsto (z, w^2)$ with image subgroup $\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z} \times \mathbb{Z}$. These subgroups are not conjugate (which is the same as being equal, in an abelian group), and so there is no choice of basepoints making the coverings isomorphic.*

There is a third covering space corresponding to the subgroup $\{(m, n) \mid m + n \text{ is even}\}$. This is the kernel of the homomorphism $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ where each generator goes to the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$. There are no other 2-sheeted covers.

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2. (a) Find all covering spaces of the circle up to covering space isomorphism.

(b) Find all homomorphisms between these covering spaces.

(c) If you ignore the covering maps to S^1, what are the covering spaces of S^1 up to homeomorphism? That is, which spaces arise in part (a)?

(A) *The subgroups of $\mathbb{Z} = \pi_1(S^1, 1)$ are $n\mathbb{Z}$ for positive integers n, and the trivial group. The universal cover is $p\colon \mathbb{R} \to S^1$ given by $t \mapsto e^{it}$. There are also coverings $p_n\colon S^1 \to S^1$ given by $e^{it} \mapsto e^{int}$ (that is, $z \mapsto z^n$). We have seen in homework that the image subgroup $p_{n*}(\pi_1(S^1, 1))$ is $n\mathbb{Z} \subset \mathbb{Z}$. Note that the coverings p_n and p_m are not isomorphic for $n \neq m$ since the subgroups are not equal (or alternatively, they have different numbers of sheets). We have found all coverings because they give all subgroups of $\mathbb{Z}$.*

(B) *First, since the trivial subgroup is contained in every subgroup, there exist homomorphisms from the universal cover to every cover. For the case $p_n\colon S^1 \to S^1$ the homomorphism $f\colon \mathbb{R} \to S^1$ is $t \mapsto e^{it/n}$, since $p_n(f(t)) = (e^{it/n})^n = e^{it}$. Next, there is a homomorphism from $p_n\colon S^1 \to S^1$ to $p_m\colon S^1 \to S^1$ if and only if $n\mathbb{Z} \subset m\mathbb{Z}$, i.e. if and only if m divides n. When this occurs, say $n = km$, the homomorphism $f\colon S^1 \to S^1$ is given by $z \mapsto z^k$, since $p_m(f(z)) = (z^k)^m = z^n = p_n(z)$.*

(C) *The covering spaces are S^1 and $\mathbb{R}$.*

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3. (a,b,c) Same as question 2, but for covering spaces of the Möbius band. The Möbius band is the quotient of the square $I \times I$ by the equivalence relation that identifies the point $(0,t)$ to the point $(1, 1-t)$ for each $t \in I$.

Here is an alternate description of the Möbius band that may be useful. Let $f\colon \mathbb{R}\times I \to \mathbb{R}\times I$ be the map $(x,t) \mapsto (x+1, 1-t)$. Then the Möbius band is the quotient of $\mathbb{R}\times I$ by the equivalence relation whose equivalence classes are the *orbits* of f. That is, $(x,t) \sim (x',t')$ if and only if $(x',t') = f^k(x,t)$ for some $k \in \mathbb{Z}$.

(A) *Let M be the Möbius band. The "core circle" $C \subset M$ (the image of $I \times \frac{1}{2}$ in the first description, or of $\mathbb{R} \times \frac{1}{2}$ in the second) is a deformation retract of M, so $\pi_1(M) \cong \mathbb{Z}$. Therefore there exist a universal cover and covers for each positive integer n.*

The universal cover is $\mathbb{R} \times I$ mapping by $(x,t) \mapsto [(x,t)]$. One can check that every point $(x,t) \in \mathbb{R} \times I$ has a neighborhood U such that $f^k(U)$ is disjoint from U for every $k \neq 0$. Then, by the definition of the equivalence relation, U maps homeomorphically onto its image in M, and the preimage of this set is simply the disjoint union $\bigcup_k f^k(U)$, with each component mapping by a homeomorphism. That is, the image of U is evenly covered by the map, and hence this is a covering space.

We know that there is a unique covering space with n sheets for every integer n. To construct it we use the fact that the covering map above factors through it, and so it is a quotient space which lies between $\mathbb{R} \times I$ and M. We define X_n to be the quotient of $\mathbb{R} \times I$ by the relation $(x,t) \sim (x',t')$ if $(x',t') = f^{kn}(x,t)$ for some $k \in \mathbb{Z}$. The covering map $p_n\colon X_n \to M$ is the natural map $[(x,t)]_n \mapsto [(x,t)]$ (remember, these are two different equivalence relations, but the former is finer than the latter, so the map is well defined). The verification that this is a covering map is similar to the argument given above.

Here is an alternate description of X_n: it is the quotient of $[0,n] \times I$ by the equivalence relation that identifies the point $(0,t)$ to the point $(n, g^n(t))$ for each $t \in I$, where $g(t) = 1 - t$. The map p_n takes the subspace $[i, i+1] \times I \subset X_n$ to M by $(x,t) \mapsto (x - i, g^i(t))$. One can check that this respects the edge-identifications and hence is well defined.

(B) *As with problem 2, there is a homomorphism $X_n \to X_m$ if and only if m divides n, and there is always a homomorphism $\mathbb{R} \times I \to X_n$. This latter homomorphism is simply given by $(x,t) \mapsto [(x,t)]_n$. For the others, if $n = km$ then the map of equivalence classes $[(x,t)]_n \mapsto [(x,t)]_m$ is well defined: if $(x,t) \sim_n (x',t')$ then $(x',t') = f^{jn}(x,t)$ for some j, hence $(x',t') = f^{(jk)m}(x,t)$, so $(x,t) \sim_m (x',t')$. It is clear that this map is compatible with the covering maps to M.*

(C) *The covering spaces are homeomorphic to M (n odd), $S^1 \times I$ (n even), and $\mathbb{R} \times I$. This is seen from the second description of X_n, which is a strip with opposite sides identified, with coherent or incoherent orientations, according to the parity of n.*

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4. Let X be the 3-fold dunce cap. That is, X is a 2-dimensional cell complex with one 0-cell, one 1-cell, and one 2-cell, whose attaching map $S^1 \to X^1$ winds 3 times around (note that X^1 is a circle). Describe a simply connected covering space of X and its covering map to X. [Hint: it is a cell complex having 3 i-cells for $i = 0, 1, 2$.]

For a picture of X, see page 52 of Hatcher. This shows a neighborhood of the 1-skeleton; just attach a disk to the boundary circle to get X. The fundamental group of X is $\mathbb{Z}/3\mathbb{Z}$. Let $p\colon \widetilde{X} \to X$

be the universal cover (which we are trying to construct). It is a 3-sheeted cover by problem 1, since the trivial subgroup has index 3 in the fundamental group.

The picture from Hatcher gives a way of understanding $\widetilde{X}$. In the part shown, if you travel once around the "equator" circle, the three "fins" will be permuted by a one-third rotation. In $\widetilde{X}$, this neighborhood of the 1-skeleton will unwind in a three-to-one fashion, and in this cover, the "fins" will make three one-third rotations before joining up again; that is, up to homeomorphism, this neighborhood is topologically a product (of the "tripod" with the circle). The disk that is missing from the picture will be covered by three disks, attached to the three boundary circles.

So $\widetilde{X}$ is a circle (made of three 0-cells and three 1-cells) with three 2-cells attached, each by a homeomorphism $S^1 \to \widetilde{X}^1$.

The map to X is given as follows. Consider $\widetilde{X}$ as three closed disks $D_1 \cup D_2 \cup D_3$ with boundary circles identified (by the identity map). Consider X as a single disk D with identifications made on its boundary (i.e. $z \sim z \cdot e^{2\pi i/3}$ for all $z \in S^1$). For $x \in D_1$ map x to $x \in D$. For $x \in D_2$ map x to $r(x) \in D$ where $r\colon D \to D$ is rotation by $2\pi/3$ (or $r(x) = x \cdot e^{2\pi i/3}$ if x is considered as a complex number). For $x \in D_3$ map x to $r^2(x) = x \cdot e^{4\pi i/3}$. This is continuous as a map $D_1 \cup D_2 \cup D_3 \to X$ and is constant on equivalence classes, hence gives a continuous map $\widetilde{X} \to X$. The interior of D is evenly covered since $r\colon D \to D$ is a homeomorphism. To check that points on the 1-skeleton have evenly covered neighborhoods, one should carefully draw pictures of these neighborhoods and their preimages in $\widetilde{X}$. Note that if there were no rotations, the "fins" of $\widetilde{X}$ would get folded together, and this folding would not be a local homeomorphism.