Exam III
Topology II
Due May 2, 2006

1. **(a)** Prove that if $p\colon (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is an n-sheeted covering space then $p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$ is a subgroup of $\pi_1(X, x_0)$ of index n.

(b) Find two 2-sheeted covering spaces of the torus $S^1 \times S^1$ that are not isomorphic to each other (as covering spaces). Can you find three?

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2. **(a)** Find all covering spaces of the circle up to covering space isomorphism.

(b) Find all homomorphisms between these covering spaces.

(c) If you ignore the covering maps to S^1, what are the covering spaces of S^1 up to homeomorphism? That is, which spaces arise in part (a)?

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3. **(a,b,c)** Same as question 2, but for covering spaces of the Möbius band. The Möbius band is the quotient of the square $I \times I$ by the equivalence relation that identifies the point $(0, t)$ to the point $(1, 1 - t)$ for each $t \in I$.

Here is an alternate description of the Möbius band that may be useful. Let $f\colon \mathbb{R} \times I \to \mathbb{R} \times I$ be the map $(x, t) \mapsto (x+1, 1-t)$. Then the Möbius band is the quotient of $\mathbb{R} \times I$ by the equivalence relation whose equivalence classes are the *orbits* of f. That is, $(x, t) \sim (x', t')$ if and only if $(x', t') = f^k(x, t)$ for some $k \in \mathbb{Z}$.

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4. Let X be the 3-fold dunce cap. That is, X is a 2-dimensional cell complex with one 0-cell, one 1-cell, and one 2-cell, whose attaching map $S^1 \to X^1$ winds 3 times around (note that X^1 is a circle). Describe a simply connected covering space of X and its covering map to X. [Hint: it is a cell complex having 3 i-cells for $i = 0, 1, 2$.]