Exam II Solutions
Topology II
March 23, 2006

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1. For any point $x_0 \in S^1$ prove that the subset $S^1 \times \{x_0\}$ is a retract of $S^1 \times S^1$, but is not a deformation retract of $S^1 \times S^1$.

The map $(x, y) \mapsto (x, x_0)$ is continuous and fixes pointwise the subspace $S^1 \times \{x_0\}$ and hence is a retraction $S^1 \times S^1 \to S^1 \times \{x_0\}$.

Deformation retractions are homotopy equivalences, and induce isomorphisms of fundamental groups. However $S^1 \times \{x_0\}$ is homeomorphic to S^1 which has fundamental group $\mathbb{Z}$, whereas $S^1 \times S^1$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$. Since these groups are not isomorphic, there can be no deformation retraction of $S^1 \times S^1$ onto $S^1 \times \{x_0\}$.

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2. **(a)** Show that if $G * H$ is abelian then G or H is trivial.

(b) Let $A = \{1, a\}$ and $B = \{1, b\}$ be groups with two elements (so $a^2 = 1$ and $b^2 = 1$). Describe all the elements of $A * B$, and describe the inverse of any element. Also, find an element of infinite order.

(A) *Suppose there exist non-trivial elements $g \in G$ and $h \in H$. Then gh and hg are reduced words in $G * H$, and are not equal words. Hence they are different as elements of $G * H$. Therefore the (length one) reduced words g and h in $G * H$ do not commute and $G * H$ is not abelian.*

(B) *The elements of $A * B$ are $\{1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \ldots\}$. That is, they are all words consisting of alternating a's and b's, including the empty word. The inverse of an element is the same word, written in reverse order. The element ab has infinite order, since $(ab)^n = abab \ldots ab$ is a reduced word not equal to 1 for any $n > 0$.*

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3. Let (X, x_0) and (Y, y_0) be spaces with basepoints and let $X \times Y$ have basepoint (x_0, y_0). Consider the standard projection maps $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$, and also the inclusion maps $i \colon X \to X \times Y$ and $j \colon Y \to X \times Y$ given by $i(x) = (x, y_0)$ and $j(y) = (x_0, y)$.

Define the homomorphism

$$\Phi \colon \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$$

by $(\gamma, \delta) \mapsto i_*(\gamma) \cdot j_*(\delta)$, where $\cdot$ denotes multiplication in $\pi_1(X \times Y, (x_0, y_0))$. Also define

$$\Psi \colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by $\gamma \mapsto (p_{1*}(\gamma), p_{2*}(\gamma))$.

Show that $\Psi \circ \Phi \colon \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is the identity.

Given $[f] \in \pi_1(X, x_0)$ and $[g] \in \pi_1(Y, y_0)$ we have

$$\begin{aligned}\Psi(\Phi([f], [g])) = \Psi(i_*([f]) \cdot j_*([g])) &= (p_{1*}(i_*([f]) \cdot j_*([g])), p_{2*}(i_*([f]) \cdot j_*([g]))) \\ &= (p_{1*}(i_*([f])) \cdot p_{1*}(j_*([g])), p_{2*}(i_*([f])) \cdot p_{2*}(j_*([g])))\end{aligned}$$

where the last equality holds because p_{1*} and p_{2*} are homomorphisms. Continuing, this element is

$$([p_1 \circ i \circ f] * [p_1 \circ j \circ g], [p_2 \circ i \circ f] * [p_2 \circ j \circ g]) = ([f] * [e_{x_0}], [e_{y_0}] * [g])$$

since $p_i \circ i = \mathrm{id}_X$, $p_i \circ j = e_{x_0}$, $p_2 \circ i = e_{y_0}$, and $p_2 \circ j = \mathrm{id}_Y$. Finally we have $([f] * [e_{x_0}], [e_{y_0}] * [g]) = ([f], [g])$ and so $\Psi \circ \Phi = \mathrm{id}$.

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4. (a) Let $h, k\colon X \to Y$ be homotopic continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$. What is the relationship between the induced homomorphisms $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $k_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_1)$? [Your answer should involve a commutative diagram.]

(b) Recall that a continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there is a continuous map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

For such maps f and g, let $f(x_0) = y_0$, $g(y_0) = x_1$, and $f(x_1) = y_1$. Show that $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ is surjective and injective.

(A) *If $H\colon X \times I \to Y$ is the homotopy from h to k then $\alpha(t) = H(x_0, t)$ is a path in Y from y_0 to y_1. The relationship between h_* and k_* is that $k_* = \hat{\alpha} \circ h_*$, where $\hat{\alpha}\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is the isomorphism induced by α. (The commutative diagram is given on page 363 of Munkres.)*

(B) *Let $k = g \circ f$ and $h = \mathrm{id}_X$. Then by (a) there is a path α from x_0 to x_1 such that $(g \circ f)_* = \hat{\alpha} \circ \mathrm{id}_* = \hat{\alpha}$ (these are homomorphisms $\pi_1(X, x_0) \to \pi_1(X, x_1)$). Since $\hat{\alpha}$ is an isomorphism and $(g \circ f)_* = g_* \circ (f_{x_0})_*$, it follows that g_* is surjective.*

Next let $k = f \circ g$ and $h = \mathrm{id}_Y$. By (a) there is a path β from y_0 to y_1 such that $(f \circ g)_* = \hat{\beta} \circ \mathrm{id}_* = \hat{\beta}$ (these are homomorphisms $\pi_1(Y, y_0) \to \pi_1(Y, y_1)$). Since $\hat{\beta}$ is an isomorphism and $(f \circ g)_* = (f_{x_1})_* \circ g_*$, it follows that g_* is injective.

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5. (a) Suppose $X = U \cup V$ where U and V are open, $U \cap V$ is path connected, $x_0 \in U \cap V$, and $i\colon U \to X$ and $j\colon V \to X$ are the inclusion maps. What can you say about $\pi_1(X, x_0)$?

(b) Let X be the space obtained by joining S^2 and S^1 at one point, and then joining another copy of S^2 to a different point of S^1. For example, let X be the union of the unit circle in the xy-plane and the spheres of radius 1 centered at the points $(\pm 2, 0, 0)$.

Prove that the fundamental group of X is cyclic (i.e. it is generated by one element). Be sure to use care when applying the theorem from part (a).

(A) *We can say that $\pi_1(X, x_0)$ is generated by the subgroups $i_*(\pi_1(U, x_0))$ and $j_*(\pi_1(V, x_0))$.*

(B) *There are many ways to show this. Here is one way in which part (a) is applied twice.*

Let A, B, and C be the sphere centered at $(-2, 0, 0)$, the unit circle, and the sphere centered at $(2, 0, 0)$ respectively. Let U be a small open neighborhood of A and let V be a small open neighborhood of $B \cup C$. Since U deformation retracts onto A, the group $\pi_1(U, x_0)$ is trivial. Since V deformation retracts onto $B \cup C$, the inclusion map $(B \cup C) \to V$ induces an isomorphism $\pi_1(B \cup C, x_0) \to \pi_1(V, x_0)$. This implies, by part (a), that $\pi_1(X, x_0)$ is generated by $i_*(\pi_1(B \cup C, x_0))$, where $i\colon B \cup C \to X$ is inclusion.

Now it suffices to show that $\pi_1(B \cup C, x_0)$ is cyclic. Let W be a small open neighborhood of B and Z a small open neighborhood of C. There are deformation retractions of W onto B and of Z onto C. Therefore $\pi_1(W, x_0)$ is infinite cyclic and $\pi_1(Z, x_0)$ is trivial. By part (a), $\pi_1(B \cup C, x_0)$ is generated by $j_*(\pi_1(W, x_0))$ (where $j\colon W \to (B \cup C)$ is inclusion) and is therefore cyclic.
