Exam II
Topology II
March 23, 2006

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1. For any point $x_0 \in S^1$ prove that the subset $S^1 \times \{x_0\}$ is a retract of $S^1 \times S^1$, but is not a deformation retract of $S^1 \times S^1$.

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2. (a) Show that if $G * H$ is abelian then G or H is trivial.

(b) Let $A = \{1, a\}$ and $B = \{1, b\}$ be groups with two elements (so $a^2 = 1$ and $b^2 = 1$). Describe all the elements of $A * B$, and describe the inverse of any element. Also, find an element of infinite order.

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3. Let (X, x_0) and (Y, y_0) be spaces with basepoints and let $X \times Y$ have basepoint (x_0, y_0). Consider the standard projection maps $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$, and also the inclusion maps $i\colon X \to X \times Y$ and $j\colon Y \to X \times Y$ given by $i(x) = (x, y_0)$ and $j(y) = (x_0, y)$.

Define the homomorphism

$$\Phi\colon \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X \times Y, (x_0, y_0))$$

by $(\gamma, \delta) \mapsto i_*(\gamma) \cdot j_*(\delta)$, where $\cdot$ denotes multiplication in $\pi_1(X \times Y, (x_0, y_0))$. Also define

$$\Psi\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by $\gamma \mapsto (p_{1*}(\gamma), p_{2*}(\gamma))$.

Show that $\Psi \circ \Phi\colon \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is the identity.

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4. (a) Let $h, k\colon X \to Y$ be homotopic continuous maps with $h(x_0) = y_0$ and $k(x_0) = y_1$. What is the relationship between the induced homomorphisms $h_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $k_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_1)$? [Your answer should involve a commutative diagram.]

(b) Recall that a continuous map $f\colon X \to Y$ is a *homotopy equivalence* if there is a continuous map $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

For such maps f and g, let $f(x_0) = y_0$, $g(y_0) = x_1$, and $f(x_1) = y_1$. Show that $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ is surjective and injective.

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5. (a) Suppose $X = U \cup V$ where U and V are open, $U \cap V$ is path connected, $x_0 \in U \cap V$, and $i\colon U \to X$ and $j\colon V \to X$ are the inclusion maps. What can you say about $\pi_1(X, x_0)$?

(b) Let X be the space obtained by joining S^2 and S^1 at one point, and then joining another copy of S^2 to a different point of S^1. For example, let X be the union of the unit circle in the xy-plane and the spheres of radius 1 centered at the points $(\pm 2, 0, 0)$.

Prove that the fundamental group of X is cyclic (i.e. it is generated by one element). Be sure to use care when applying the theorem from part (a).
