Exam I Solutions
Topology II
February 21, 2006

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1. (a) If $r\colon X \to A$ is a retraction, what can you say about the map $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$, where $i\colon A \hookrightarrow X$ is inclusion and $a_0 \in A$? Give a proof of your statement.

(b) Show that the fundamental group of the "figure eight" is infinite, by using a retraction to a subspace. [The figure eight is the union of two circles that touch in one point.]

(A) *The homomorphism $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective. We have $r \circ i = \mathrm{id}_A$ so apply π_1 to obtain: $r_* \circ i_* = \mathrm{id}$. Since* id *is injective, i_* must also be injective.*

(B) *Consider X as two circles joined together, called* left *and* right*. Let $A \subset X$ be the left circle. Let $r\colon X \to A$ be the map which is the identity on A, and which maps the right circle to a_0. This is continuous by the pasting lemma and so A is a retract of X. Hence $\pi_1(X, x_0)$ contains $\pi_1(A, x_0) \cong \mathbb{Z}$ as a subgroup, and hence is infinite.*

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2. (a) State the Tietze extension theorem.

(b) Let Z be a space which is a union of two closed sets $X \cup Y$, where X and Y are normal. Show that Z is normal.

(A) *Tietze extension theorem: let X be a normal space and let $A \subset X$ be a closed subset. Any continuous map $f\colon A \to [a, b]$ extends to a continuous map of X to $[a, b]$. The same statement also holds with $[a, b]$ replaced by $\mathbb{R}$.*

(B) *Let $A, B \subset Z$ be disjoint closed sets. It suffices to show that there is a continuous function $f\colon Z \to [0, 1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.*

Since $A \cap X$ and $B \cap X$ are closed in X, there is a continuous function $f_1\colon X \to [0, 1]$ which is 0 on $A \cap X$ and 1 on $B \cap X$. (Use the Urysohn Lemma, or Tietze.) By the pasting lemma, the function $f_2\colon X \cup A \cup B \to [0, 1]$ which is 0 on A, 1 on B, and f_1 on X, is continuous.

The subset $C = (X \cup A \cup B) \cap Y$ is closed in Y, so the map $f_2|_C\colon C \to [0, 1]$ extends to $f_3\colon Y \to [0, 1]$, by Tietze. By the pasting lemma, f_2 and f_3 together define a continuous function $f\colon Z \to [0, 1]$ which is 0 on A and 1 on B.

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3. Let X_0 be a path component of X and let $x_0 \in X_0$ be a basepoint. Show that the inclusion map $X_0 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

If $f\colon I \to X$ is a loop at x_0 then since X_0 is path connected, the image of f is in X_0. Therefore f may be written as $i \circ f'$ where $i\colon X_0 \to X$ is inclusion and f' is the map f with restricted range X_0. Now $[f] = [i \circ f'] = i_*([f'])$ and so $i_*\colon \pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is surjective.

If $i_*([f]) = 1$ in $\pi_1(X, x_0)$ then $(i \circ f) \simeq_p e_{x_0}$ in X via a path homotopy $F\colon I \times I \to X$. Since $I \times I$ is path connected, $F(I \times I) \subset X_0$. Restricting the range of F gives a path homotopy

$F'\colon I \times I \to X_0$ between f and e_{x_0}, and so $f \simeq_p e_{x_0}$ in X_0. Hence $[f] = 1$ in $\pi_1(X_0, x_0)$ and i_* is injective.

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4. (a) State the Borsuk-Ulam theorem for S^2.

(b) Suppose that the sphere S^2 is expressed as a union of three closed sets: $S^2 = A_1 \cup A_2 \cup A_3$. Show that one of the sets A_i contains an antipodal pair $\{x, -x\}$. [Hint: use the functions $f_i(x) = \operatorname{dist}(x, A_i)$ for $i = 1, 2$.]

(A) *If $f\colon S^2 \to \mathbb{R}^2$ is continuous then there is a point $x \in S^2$ such that $f(x) = f(-x)$.*

(B) *Let $f_i(x) = \operatorname{dist}(x, A_i)$ for $i = 1, 2$. Note that since A_i is compact, $x \in A_i$ if and only if $f_i(x) = 0$. The function $f = f_1 \times f_2$ is a continuous map $S^2 \to \mathbb{R}^2$. Hence there is a point $x \in S^2$ such that $f_1(x) = f_1(-x)$ and $f_2(x) = f_2(-x)$.*

If $f_1(x) = 0$ then $f_1(-x) = 0$ and $\{x, -x\} \subset A_1$. If $f_2(x) = 0$ then $f_2(-x) = 0$ and $\{x, -x\} \subset A_2$. Otherwise $f_1(x)$, $f_2(x)$, $f_1(-x)$, and $f_2(-x)$ are positive. Since $S^2 = A_1 \cup A_2 \cup A_3$, we must have $\{x, -x\} \subset A_3$.

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5. Let $p\colon E \to B$ be a covering map.

(a) Show that if B is Hausdorff then so is E.

(b) Suppose $p(e_0) = b_0$. Show that the induced homomorphism $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. State clearly any theorems that you use.

(A) *Let $x, y \in E$ be distinct points. If $p(x) = p(y)$ let $U \subset B$ be an evenly covered neighborhood of $p(x)$. Then x and y are in distinct slices V_x and V_y each mapping homeomorphically to U by p, and these open sets are disjoint.*

If $p(x) \neq p(y)$ then since Y is Hausdorff, there are disjoint open sets $U, V \subset Y$ such that $p(x) \in U$ and $p(y) \in V$. Then $p^{-1}(U)$ and $p^{-1}(V)$ separate x and y.

(B) *Homotopy Lifting Lemma: If $F\colon I \times I \to B$ is a path homotopy with $F(0,0) = b_0$ then there is a unique lift $\widetilde{F}\colon I \times I \to E$ such that $\widetilde{F}(0,0) = e_0$. Moreover, $\widetilde{F}$ is a path homotopy.*

If $p_*([f]) = 1$ in $\pi_1(B, b_0)$ then the loop $p \circ f\colon I \to B$ is path homotopic to the constant loop e_{b_0}. Let F be such a path homotopy. Then there is a path homotopy $\widetilde{F}\colon I \times I \to E$ with $\widetilde{F}(0,0) = e_0$. This path homotopy is a path homotopy from f to the constant map e_{e_0} since the restrictions of $\widetilde{F}$ to $I \times \{0\}$ and $I \times \{1\}$ are the unique lifts of the paths $p \circ f$ and e_{b_0} starting at e_0. Hence $[f] = 1$ in $\pi_1(E, e_0)$ and p_* is injective.

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6. (a) Show that if Y is Hausdorff then the space of continuous maps $\mathscr{C}(X, Y)$ with the compact-open topology is Hausdorff.

(b) Consider the sequence of functions $f_n \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ given by $f_n(x) = x/n$. Does this sequence converge in the compact-open topology? Explain why or why not.

(A) *Take $f, g \in \mathscr{C}(X, Y)$ which are not equal. Then $f(x) \neq g(x)$ for some $x \in X$. Let U, V be disjoint open sets in Y with $f(x) \in U$ and $g(x) \in V$. Hence $f \in S(\{x\}, U)$ and $g \in S(\{x\}, V)$. There open sets in $\mathscr{C}(X, Y)$ are disjoint because U and V are, and x cannot map into both.*

(B) *We claim that* $f_n \to 0$. *Let* $S(C_1, U_1) \cap \cdots \cap S(C_n, U_n)$ *be an open neighborhood of the function* 0. *Hence each* U_i *contains* 0. *Let* $C = \bigcup_i C_i$ and $U = \bigcap_i U_i$. *Then* $S(C, U) \subset S(C_1, U_1) \cap \cdots \cap S(C_n, U_n)$, *and it suffices to show that* $f_n \in S(C, U)$ *for almost all* n.

Choose $K, \epsilon > 0$ *such that* $|x| < K$ *for all* $x \in C$ and U *contains* $(-\epsilon, \epsilon)$. *Then* $n > K/\epsilon$ *implies* $|f_n(x)| = |x/n| < K/n < \epsilon$ *for all* $x \in C$, *and therefore* $f_n \in S(C, U)$ *for all such* n.
