Exam I
Topology II
February 21, 2006

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1. **(a)** If $r\colon X \to A$ is a retraction, what can you say about the map $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$, where $i\colon A \hookrightarrow X$ is inclusion and $a_0 \in A$? Give a proof of your statement.

(b) Show that the fundamental group of the "figure eight" is infinite, by using a retraction to a subspace. [The figure eight is the union of two circles that touch in one point.]

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2. **(a)** State the Tietze extension theorem.

(b) Let Z be a space which is a union of two closed sets $X \cup Y$, where X and Y are normal. Show that Z is normal.

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3. Let X_0 be a path component of X and let $x_0 \in X_0$ be a basepoint. Show that the inclusion map $X_0 \hookrightarrow X$ induces an isomorphism of fundamental groups $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

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4. **(a)** State the Borsuk-Ulam theorem for S^2.

(b) Suppose that the sphere S^2 is expressed as a union of three closed sets: $S^2 = A_1 \cup A_2 \cup A_3$. Show that one of the sets A_i contains an antipodal pair $\{x, -x\}$. [Hint: use the functions $f_i(x) = \mathrm{dist}(x, A_i)$ for $i = 1, 2$.]

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5. Let $p\colon E \to B$ be a covering map.

(a) Show that if B is Hausdorff then so is E.
(b) Suppose $p(e_0) = b_0$. Show that the induced homomorphism $p_*\colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. State clearly any theorems that you use.

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6. **(a)** Show that if Y is Hausdorff then the space of continuous maps $\mathscr{C}(X, Y)$ with the compact-open topology is Hausdorff.

(b) Consider the sequence of functions $f_n \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ given by $f_n(x) = x/n$. Does this sequence converge in the compact-open topology? Explain why or why not.
